Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

Spring 2022 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

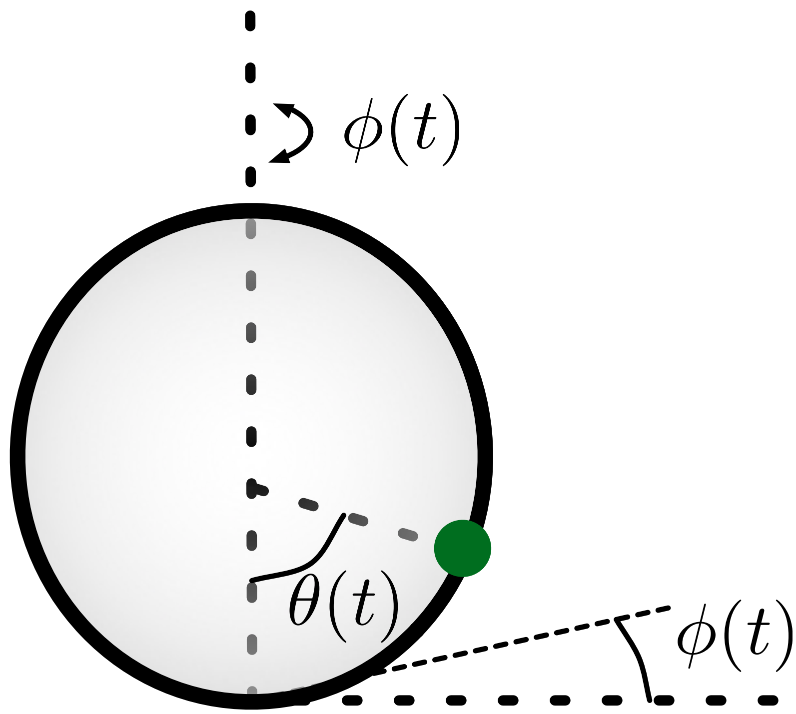
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A bead on a driven ring

A small bead slides freely on a ring of radius a in the earths gravitational field. The face of the ring is at an angle $\phi(t)$ with respect to the x-axis, and is driven harmonically with small amplitude ϕ_0 and rather high frequency ω, i.e. $\phi(t) = \phi_0 \cos(\omega t)$.



(a) (6 points) Determine the Lagrangian of the system without approximation, and find the equations of motion.

(b) (4 points) Determine the (effective) equation of motion for the bead on a time scale which is long compared to $1/\omega$.

Hint: Assume that θ is approximately constant over the time scale set by $1/\omega$, and average the equation of motion over the rapid oscillations. Neglect terms of order ϕ_0^4.

(c) (6 points) Consider a bead at the bottom of the ring. Show that this configuration is unstable when the frequency is greater than ω_c. Determine ω_c.

(d) (4 points) Determine the steady state position of the bead for $\omega > \omega_c$. Take the limit of your result for asymptotically large ω, and describe the result physically.

Solution:

(a) We have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz\,. \tag{1}$$

Taking the usual coordinate system

$$\begin{aligned} z =& -a\cos\theta\,, &(2)\\ x =& a\sin\theta\cos\phi\,, &(3)\\ y =& a\sin\theta\sin\phi\,, &(4) \end{aligned}$$

we find

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mga\cos(\theta)\,. \tag{5}$$

Using $\phi = \phi_0\cos(\omega t)$ and $\sin^2\omega t = \frac{1}{2} - \frac{1}{2}\cos(2\omega t)$ we find

$$\begin{aligned} L =& \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2\phi_0^2\ \sin^2\theta\sin^2(\omega t)) + mga\cos(\theta)\,, &(6)\\ =& \frac{1}{2}(ma^2)\dot{\theta}^2 - V_0(\theta) - V_1(\theta)\cos(2\omega t)\,. &(7)\\ & &(8) \end{aligned}$$

Here

$$V_0 = -mga\cos\theta - \frac{1}{4}(ma^2)(\omega\phi_0)^2\sin^2\theta\,, \tag{9}$$

and

$$V_1 = \frac{1}{4}(ma^2)(\omega\phi_0)^2\sin^2\theta\,. \tag{10}$$

The equation of motion take the form

$$(ma^2)\ddot{\theta} = -\frac{\partial V_0}{\partial\theta} - \frac{\partial V_1}{\partial\theta}\cos(2\omega t)\,. \tag{11}$$

(b) We can simply average over the oscillations to find the mean equations of motion

$$\begin{aligned} (ma^2)\ddot{\theta} =& -\frac{\partial V_0}{\partial\theta} - \overline{\frac{\partial V_1}{\partial\theta}\cos(2\omega t)}\,, &(12)\\ \simeq& -\frac{\partial V_0}{\partial\theta}\,. &(13) \end{aligned}$$

Here we have neglected the so called ponderamotive potential is

$$U_{\text{eff}}(\theta) = \frac{1}{4(ma^2)(2\omega)^2}\left(\frac{\partial V_1}{\partial\theta}\right)^2\,, \tag{14}$$

which is of order ϕ_0^4.

(c) The effective equation of motion takes the form

$$(ma^2)\ddot{\theta} = -\frac{\partial V_0}{\partial \theta}\,. \tag{15}$$

The minimum of V_0 is at $\theta = 0$ provided the frequency is not too large. Expanding V_0 at θ zero gives

$$V_0 = -mga + \theta^2 \left[\frac{mga}{2} - \frac{(ma^2)(\omega\phi_0)^2}{4} \right] + \ldots\,. \tag{16}$$

Whenever the term in square brackets is negative, $\theta = 0$ will be a maximum and not a minimum. The critical frequency is

$$\omega_c^2 = 2\left(\frac{g}{a\phi_0^2}\right) \tag{17}$$

(d) Differentiating we have

$$\frac{\partial V_0}{\partial \theta} = 0\,, \tag{18}$$

we find

$$mg\sin\theta - \frac{1}{2}ma^2(\omega\phi_0^2)\sin\theta\cos\theta = 0\,. \tag{19}$$

So we find finally

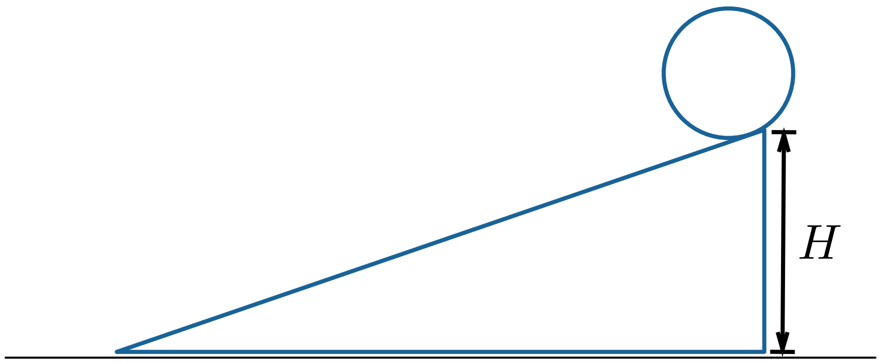
$$\cos\theta = \frac{\omega_c^2}{\omega^2}\,, \tag{20}$$

which clearly only makes sense for $\omega > \omega_c$. For $\omega \gg \omega_c$ the $\cos\theta = 0$ and $\theta \simeq \pi/2$. Clearly, this should be the case as the centrifugal force becomes very large for $\omega \to \infty$.

Classical Mechanics 2

A cylinder on a sliding ramp

A solid cylinder of mass m and radius R starts from rest at height H, and rolls without slipping down a ramp of mass $m_0 = 2m$. The ramp slides without friction to the right in response to the motion of the cylinder.



(a) (4 points) Write down a set of coordinates to describe the motion of the cylinder and the ramp. Write down a Lagrangian of the system using these coordinates.

(b) (8 points) Find the angular velocity of the cylinder and the velocity ramp just before the cylinder reaches the bottom.

(c) (8 points) Determine the normal force between the cylinder and the ramp.

Hint: Use a Lagrange multiplier to enforce the condition that the cylinder remains on the slope.

Solution:

(a) The base of the cylinder at x_0. The center of mass cooridnates of the cylinder are

$$(X, Y) = (x_0 + R\phi\cos\theta + \text{const}, R\phi\sin\theta + \text{const}) \tag{1}$$

The kinetic energy is the velocity of the center of mass plus the rolling energy.

$$(\dot{X}, \dot{Y}) = (\dot{x}_0 + R\dot{\phi}\cos\theta, R\dot{\phi}\sin\theta) \tag{2}$$

The Lagrangian is

$$L = \frac{1}{2}(2m)\dot{x}_0^2 + \frac{1}{2}m(\dot{x}_0 + R\dot{\phi}\cos\theta)^2 + \frac{1}{2}m(R\dot{\phi}\sin\phi)^2 + \frac{1}{2}I\dot{\phi}^2 - mgR\phi\sin\theta \tag{3}$$

$$= \frac{1}{2}3m\dot{x}_0^2 + m\dot{x}_0R\dot{\phi}\cos\theta + \frac{1}{2}(mR^2 + I)\dot{\phi}^2 - mgR\phi\sin\theta \tag{4}$$

where $I = 1/2mR^2$.

(b) The x_0 variable is cyclic leading to the conservation law

$$2m\dot{x}_0 + m(\dot{x}_0 + R\dot{\phi}\cos\theta) = P_0 \tag{5}$$

which is clearly identified as the total x momentum of the system. In the current setup, where $P_0 = 0$, we have

$$-\frac{R\cos\theta\dot{\phi}}{3} = \dot{x}_0 \tag{6}$$

The energy is conserved as well leading

$$E = \frac{1}{2}3m\dot{x}_0^2 + m\dot{x}_0R\dot{\phi}\cos\theta + \frac{1}{2}(mR^2 + I)\dot{\phi}^2 + mgR\phi\sin\theta \tag{7}$$

$$= \frac{1}{6}m\cos^2\theta(R\dot{\phi})^2 - \frac{1}{3}m(R\dot{\phi})^2\cos^2\theta + \frac{3}{4}m(R\dot{\phi})^2 + mgR\phi\sin\theta \tag{8}$$

$$= \frac{1}{12}m\left(9 - 2\cos^2\theta\right)(R\dot{\phi})^2 + mgR\phi\sin\theta \tag{9}$$

The total value of the energy is $E = mgH$ and at the bottom the ϕ is zero, yielding

$$R\dot{\phi} = -\sqrt{\frac{12gH}{9 - 2\cos^2\theta}} \tag{10}$$

We have the speed of the ramp which is

$$\dot{x}_0 = \frac{\cos\theta}{3}\sqrt{\frac{12gH}{9 - 2\cos^2\theta}} \tag{11}$$

(c) We can use the method of multipliers or other means. The Lagrangian is

$$L = \frac{1}{2}(2m)\dot{x}_0^2 + \frac{1}{2}m(\dot{x}_0 + R\dot{\phi}\cos\theta)^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 - mgy + \lambda(y - R\phi\sin\theta) \tag{12}$$

Finding the equation of motion we have

$$
\begin{aligned}
3m\ddot{x}_0 + mR\ddot{\phi}\cos\phi =& 0 && (13)\\
m(\ddot{x}_0 + R\ddot{\phi}\cos\theta)R\cos\theta + \frac{1}{2}mR^2\ddot{\phi} =& -\lambda R\sin\theta && (14)\\
m\ddot{y} =& -mg + \lambda && (15)\\
y =& R\phi\sin\theta && (16)
\end{aligned}
$$

Combining the equations we use $\ddot{x}_0 = -R\cos\theta\ddot{\phi}/3$, and $R\ddot{\phi} = (-g + \lambda/m)/\sin\theta$ So

$$
mR^2\ddot{\phi}\left(\frac{2}{3}\cos^2\theta + \frac{1}{2}\right) = -\lambda R\sin\theta \tag{17}
$$

Or

$$
(mg - \lambda)\frac{(\frac{2}{3}\cos^2\theta + \frac{1}{2})}{\sin^2\theta} = \lambda \tag{18}
$$

So solving for λ

$$
\lambda = \frac{mgu}{1+u} \qquad u \equiv \frac{\frac{2}{3}\cos^2\theta + \frac{1}{2}}{\sin^2\theta} \tag{19}
$$

The Lagrange multiplies is the $N\cos\theta$ (as is clear from Eq. 15) leading finally to

$$
N = \frac{1}{\cos\theta}\left(\frac{mgu}{1+u}\right) \tag{20}
$$

Classical Mechanics 3

Ion trapping

Consider a positively charged particle of charge q and mass m is placed in a electrostatic potential

$$\Phi(x, y, z) = \frac{V_0}{L^2}\left(z^2 - \tfrac{1}{2}\rho^2\right) , \tag{1}$$

where $\rho = \sqrt{x^2 + y^2}$ is the cylindrical radius. There is a magnetic-field of magnitude B_0 in the z direction.

(a) (1 point) Without the magnetic field, explain why the motion of the charged particle near the origin is unstable, i.e. the radius increases without bound.

(b) (5 points) Now include the magnetic field. Write down the Lagrangian of the system.

Hint: For this problem it is convenient to take the gauge $\boldsymbol{A} = \frac{1}{2}B_0(-y, x, 0)$. Parametrize the strength the magnetic field by the cyclotron frequency $\omega_B \equiv qB/mc$.

(c) (2 points)Show the equation of motion for the z coordinate is harmonic, and determine the oscillation frequency ω_z. Show from the equation of motion that the "z energy" (i.e. $\mathcal{E}_z = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\omega_z^2 z^2$) is constant.

(d) (3 points) Determine the other integrals (or constants) of the motion.

(e) (5 points) For a sufficiently strong magnetic field $B_0 > B_{\rm crit}$, the particle's motion is bounded between $\rho_{\rm min}$ and $\rho_{\rm max}$:

(i) Determine $B_{\rm crit}$.

(ii) Find $\rho_{\rm min}$ and $\rho_{\rm max}$ in terms of ω_z, ω_B, and the integrals of motion.

Hint: Use energy considerations to analyze the motion.

(f) (4 points) Determine the fixed radius ρ_0 of circularly cylindrical orbits. Find the angular velocity ω_0 of these orbits in terms of the integrals of the motion. Answer the following:

(i) Is the motion clockwise or counter-clockwise when viewed from above?

(ii) For $B_0 \gg B_{\rm crit}$, describe qualitatively the circularly cylindrical motion of the trapped charged particles.

Solution:

(a) This is clear the potential is a saddle point $U(x, y, z) = q\varphi(x, y, z)$ and not a minimum.

(b) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{qV_0}{L^2}\left(z^2 + \frac{1}{2}(x^2 + y^2)\right) - \frac{qB_0}{2mc}(\dot{x}y - \dot{y}x) \tag{2}$$

$$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - \frac{qV_0}{L^2}\left(z^2 - \frac{1}{2}\rho^2\right) + \frac{1}{2}m\omega_B\rho^2\dot{\phi} \tag{3}$$

(c) The equation of motion is

$$m\ddot{z} = -\frac{2qV_0}{L^2}z \tag{4}$$

So the associated oscillation frequency is

$$\omega_z^2 = \frac{2qV_0}{mL^2} \tag{5}$$

To show that energy is constant we use the elementary argument

$$\ddot{z} = \frac{dv_z}{dt} \tag{6}$$

We multiply both sides by v_z yield

$$\frac{1}{2}m\frac{d}{dt}v_z^2 = -m\omega_z^2\frac{d}{dt}z^2 \tag{7}$$

So we find

$$\frac{1}{2}mv_z^2 + \frac{1}{2}m\omega_z^2 z^2 = \text{const} \tag{8}$$

(d) There is the total energy

$$E = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}m\omega_z^2\left(z^2 - \frac{1}{2}\rho^2\right) \tag{9}$$

and the angular momentum

$$p_\phi = m\rho^2\dot{\phi} + \frac{1}{2}m\omega_B\rho^2 \tag{10}$$

The energy can be written

$$E = \mathcal{E}_z + \frac{1}{2}m\dot{\rho}^2 + \frac{\left(p_\phi - \frac{1}{2}m\omega_B\rho^2\right)^2}{2m\rho^2} - \frac{1}{4}m\omega_z^2\rho^2 \tag{11}$$

$$E = \mathcal{E}_z - \tfrac{1}{2}p_\phi\omega_B + \frac{1}{2}m\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{8}m(\omega_B^2 - 2\omega_z^2)\rho^2 \tag{12}$$

(e) Define $\epsilon \equiv E - E_z + \frac{1}{2}p_\phi \omega_B$. Then energy conservation is

$$\epsilon = \frac{1}{2}m\dot{\rho}^2 + V_{\text{eff}}(\rho) \tag{13}$$

where

$$V_{\text{eff}}(\rho) = \frac{p_\phi^2}{2m\rho^2} + \frac{1}{8}m\left(\omega_B^2 - 2\omega_z^2\right)\rho^2 \tag{14}$$

The potential is concave up if $\omega_B > \sqrt{2}\omega_z$. The turning points are when $\dot{\rho} = 0$

$$\epsilon = \frac{p_\phi^2}{2m\rho^2} + \frac{1}{8}m\Delta\omega^2\rho^2 \tag{15}$$

Which is a quadratic equation for ρ^2

$$m\Delta\omega^2(\rho^2)^2 - 2\epsilon\rho^2 + \frac{p_\phi^2}{4m} = 0 \tag{16}$$

So

$$\rho^2 = \frac{1}{m\Delta\omega^2}\left[\epsilon \pm \sqrt{\epsilon^2 - (\tfrac{1}{2}\Delta\omega p_\phi)^2}\right] \tag{17}$$

(f) The circular orbits are at the minimum of the effective potential

$$-\frac{p_\phi^2}{m^2\rho_0^3} + \frac{1}{4}m\Delta\omega^2\rho_{=}0 \tag{18}$$

So

$$\rho_0 = \left(\frac{2p_\phi}{m\Delta\omega}\right)^{1/2} \tag{19}$$

The angular velocity is

$$\omega_0 = \dot{\phi} = \frac{p_\phi}{m\rho^2} - \frac{1}{2}\omega_B \tag{20}$$

$$= \frac{1}{2}\left(\omega_B^2 - 2\omega_z^2\right)^{1/2} - \frac{1}{2}\omega_B \tag{21}$$

$$\simeq -\frac{\omega_z^2}{2\omega_B} \tag{22}$$

So the motion is clockwise ($\dot{\phi}$ is negative) . We have used here that $\omega_B \gg \omega_z$, which follows since $B_0 \gg B_{\text{crit}}$. Since $\omega_0 \ll \omega_z$ in magnitude. We have periodic motion in the z direction, which is accompanied by slow clockwise precession of the circular orbit.

Electromagnetism 1

Counter rotations of a charged sphere

Consider a stationary sphere of radius R. The sphere consists of two hemispheres that are held together on the xy plane. The electric potential inside the sphere is given

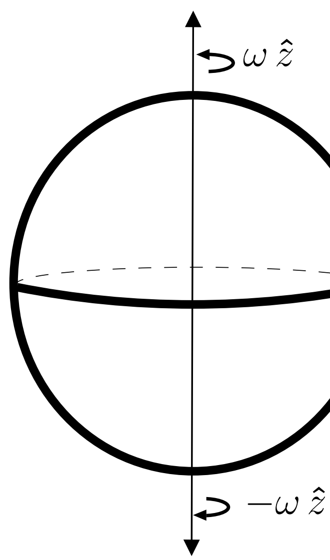
$$\phi = Cr^3 \cos\theta \,, \tag{1}$$

where C is a positive normalizing constant, $r \equiv \sqrt{x^2 + y^2 + z^2}$, and θ is the polar angle.

(a) (4 points) Find the charge density inside the sphere.

(b) (5 points) Find the leading order potential and electric field far from the sphere. Qulatitatively sketch the electric field lines.

Now the two hemispheres are set to counter-rotate with a constant angular velocity ω around the z axis, i.e. the top hemisphere rotates with $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$, and the bottom hemisphere rotates at $\boldsymbol{\omega} = -\omega\hat{\boldsymbol{z}}$.



(c) (6 points) Determine the magnetic vector potential (in the Coulomb gauge) far from the sphere.

Finally, the two hemispheres are set to counter rotate as in the previous item, but with a time dependent angular velocity $\omega(t) = \omega_0 \cos(\Omega t)$, with Ω small.

(d) (5 points) Find the induced electric to first order in Ω. (i) Over what range in radius is your result a small correction to (b)? (ii) Qualitatively sketch the electric field lines associated with this correction.

Solution:

(a) We write $\phi = C(x^2 + y^2 + z^2)z$. Evaluating the Laplacian we have

$$\frac{\rho}{\epsilon_0} = -\nabla^2\phi = (2C + 2C + 6C)z = -10Cz \tag{2}$$

(b) The system clearly has an electric dipole pointing along $\hat{z}$. Evaluating the dipole moment

$$\boldsymbol{p} = \left[\int_{\text{sphere}} \rho(z) z \, dV\right] \hat{\boldsymbol{z}} \tag{3}$$

Noting that $\int z^2 \, dV = \frac{1}{3}\int r^2 \, dV$ we find

$$p_z = -\frac{10C\epsilon_0}{3}\int_0^R r^2 \,(4\pi r^2 dr) \tag{4}$$

$$= -\frac{2}{3}(4\pi\epsilon_0)CR^5 \tag{5}$$

Generally the potential outside the sphere takes the form

$$\phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \tag{6}$$

In particular, the $\ell = 1$ terms have the following form

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{p_z \cos\theta}{r^2} - \mathcal{E}_z r\cos\theta \tag{7}$$

where p_z is determined by B_1, and $\mathcal{E}_z$ is determined by A_1.

We demand continuity of the electric field at the surface

$$E_r = -\left.\frac{\partial\phi_r}{\partial r}\right|_{r=R} \tag{8}$$

$$E_\theta = -\left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=R} \tag{9}$$

From the potential inside the sphere we have

$$E_r = -3CR^2\cos\theta \tag{10}$$

$$E_\theta = CR^3\sin\theta \tag{11}$$

From the potential outside the sphere

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{2p_z\cos\theta}{R^3} + \mathcal{E}_z\cos\theta \tag{12}$$

$$E_\theta = \frac{1}{4\pi\epsilon_0}\frac{p_z\sin\theta}{R^3} - \mathcal{E}_z\sin\theta \tag{13}$$

Comparing these terms we have

$$p_z = -\frac{2}{3}(4\pi\epsilon_0)\, CR^5 \qquad \mathcal{E}_z = -\frac{5}{3}CR^2 \tag{14}$$

So to summarize, the electric field outside the sphere involves the dipole field and a constant electric field in the negative $\boldsymbol{z}$ direction

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{3\hat{\boldsymbol{r}}(\boldsymbol{p}\cdot\hat{\boldsymbol{r}}) - \boldsymbol{p}}{r^3} + \mathcal{E}_z\hat{\boldsymbol{z}} \tag{15}$$

where the dipole moment is $\boldsymbol{p} = p_z\hat{\boldsymbol{z}}$ and both p_z and $\mathcal{E}_z$ are negative in our conventions, and are given by Eq. (14)

(c) The dipole term in the vector potential is given by

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{n}}{r^2}, \tag{16}$$

where $\boldsymbol{m}$ is the magnetic dipole moment

$$\boldsymbol{m} = \frac{1}{2}\int(\boldsymbol{x'}\times\boldsymbol{J})dV \tag{17}$$

In spherical coordinates, taking θ as polar angle, current density is given by

$$\begin{aligned} \boldsymbol{J} &= \rho\boldsymbol{v} && (18)\\ &= -10C\epsilon_0\omega r^2\sin\theta|\cos\theta|\hat{\phi}. && (19) \end{aligned}$$

Noting that $\boldsymbol{x'} = r\hat{r}$ in spherical coordinates and $\hat{r}\times\hat{\phi} = -\hat{\theta}$, and $dV = r^2\sin\theta dr d\theta d\phi$, and decomposing $\hat{\theta}$ in terms of Cartesian components ($\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$), the expression for $\boldsymbol{m}$ can be directly integrated to give

$$\boldsymbol{m} = -\frac{5}{6}\pi C\epsilon_0\omega R^6\hat{z} \tag{20}$$

Substitute $\boldsymbol{m}$ in Eqn. 16 (which is evaluted using the right-hand rule)

$$\boldsymbol{A} = -\frac{5}{24c^2}C\frac{R^6}{r^2}\omega\sin\theta\,\hat{\boldsymbol{\phi}} \tag{21}$$

where we used $c^2 = 1/\epsilon_0\mu_0$. The intuition here is that the vector potential follows the currents, which are flowing in the negative azimuthal direction.

(d) We can consider the source terms for $\boldsymbol{E} = -\nabla\phi - \partial_t A$ as a superposition of two effects: the charge distribution, which does not change in time, resulting in ϕ as expressed in part (b) & the induced electric field, $\boldsymbol{E}_{\text{ind}}$, which constitutes the correction to the result of part (b).

$$\begin{aligned} \boldsymbol{E}_{\text{ind}} &= -\partial_t\boldsymbol{A} && (22)\\ &= \frac{5}{24c^2}C\frac{R^6}{r^2}\frac{d\omega}{dt}\sin\theta\hat{\boldsymbol{\phi}} && (23)\\ &= -\frac{5}{24c^2}C\frac{R^6}{r^2}\omega_0\Omega\sin(\Omega t)\sin\theta\hat{\boldsymbol{\phi}} && (24) \end{aligned}$$

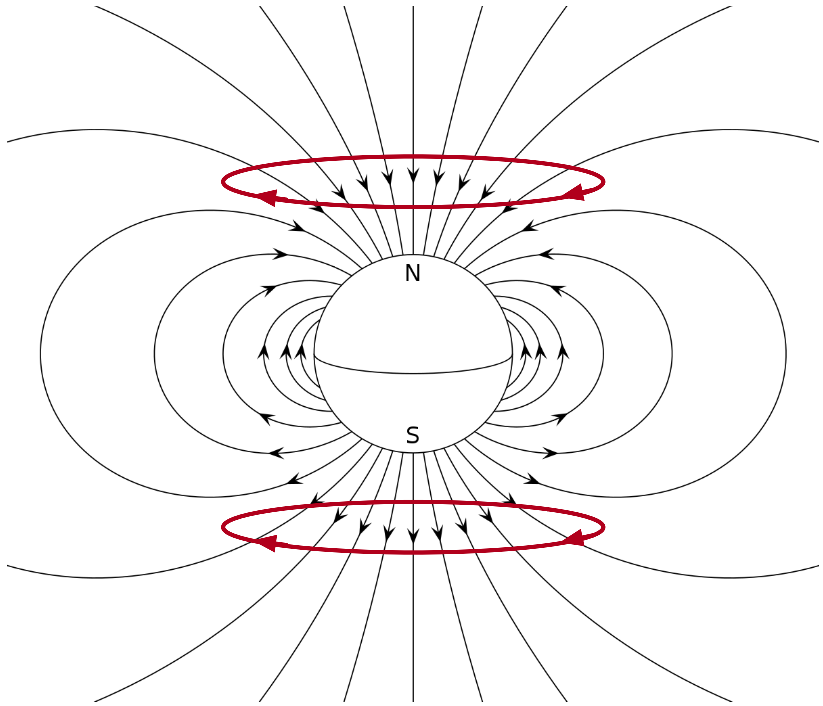


Figure 1: Electric field lines at $t = 0$ due to the sphere. At $t = 0$ there is an electrostatic dipole field. We also note that the magnetic field lines are qualitatively similar to the dipole electric field, as the sphere has a magnetic dipole moment in the negative z direction. The induced fields (shown in red) are in the azimuthal direction, and have a qualitatively different character. At $t = 0$ the current flow due to the motion of the sphere is clockwise (when viewed from above) and is decreasing in magnitude. The induced electric field tries to prevent this waning current and is therefore also clockwise. One can also use Lenz' law – the magnetic flux through the red loops is decreasing, and the displacement currents (the induced electric field) tries to reenforce the magnetic field.

The electrostatic field from part (b) is

$$\boldsymbol{E}_{\mathbf{ES}} = -\nabla\phi \tag{25}$$

$$= -\frac{2}{3}C\frac{R^5}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{26}$$

A figure showing the field lines is shown in Fig. 1 Comparing the amplitude scaling of the two components,

$$\boldsymbol{E}_{\mathbf{ind}} \ll \boldsymbol{E}_{\mathbf{ES}} \tag{27}$$

$$\frac{R^6}{r^2c^2}\omega_0\Omega \ll \frac{R^5}{r^3} \tag{28}$$

$$r \ll \frac{c^2}{R\omega_0\Omega} \tag{29}$$

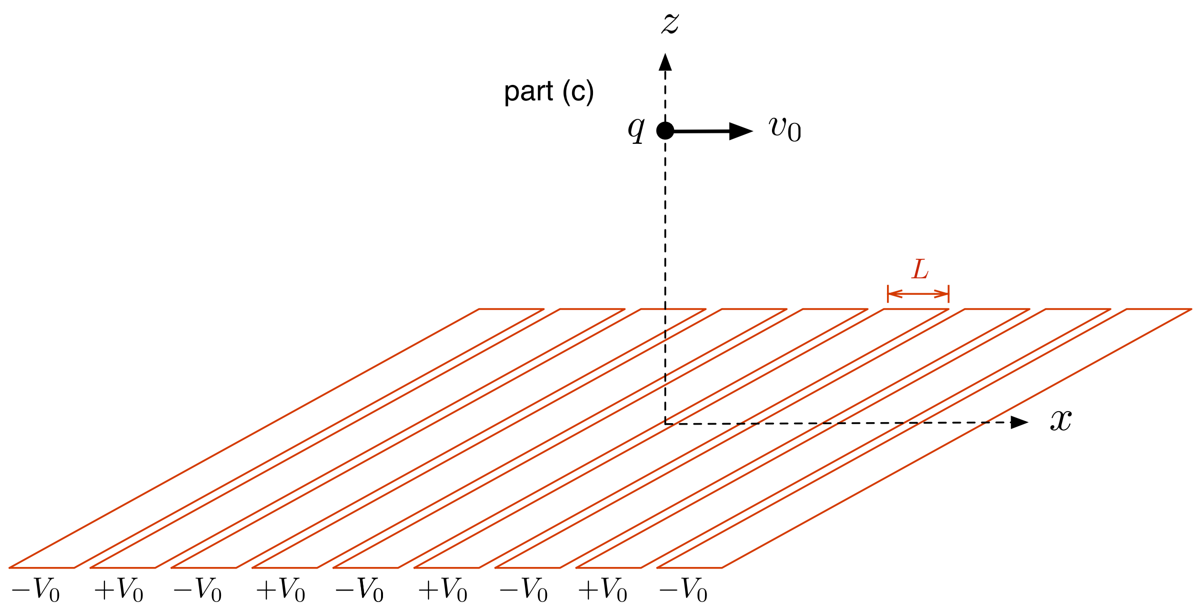
Thus we see that the induced electric field is small compared to the electrostatic field provided we don't look too far away from the source. We require that the length is smaller than a characteristic length L_0, $r \ll L_0$. This characteristic length L_0 is set by a combination of the wavelength $\lambda \sim c/\Omega$ of the emitted radiation, and the nonrelativistic character of the source which has v/c where $v = R\omega_0$, which naturally makes magnetic effects small compared to the electric ones. The characteristic length is of order

$$L_0 \sim \frac{\lambda}{(v/c)} \tag{30}$$

Electromagnetism 2

Strips with a boost

Consider an infinite set of long metal strips filling the xy plane as shown below. The strips have width L and the spacing between the strips is negligible. The strips are held at alternating potentials $\pm V_0$ as shown below.



(a) (7 points) Determine the potential $\phi(x, z)$ everywhere above the xy plane.

Hint: Use separation of variables to show that the solution takes the form of a series expansion:

$$\phi(x, z) = \sum_k (A_k \cos(kx) + B_k \sin(kx)) \, Z_k(z) \,. \tag{1}$$

Determine the allowed values of k, the functional form of $Z_k(z)$, and the coefficients, A_k and B_k.

(b) (2 points) For $z \gg L$, determine the dominant term in the expansion developed in (a). Explain your reasoning with a sentence or two.

Now consider a charge particle of mass m and charge q which moves relativistically along the x-axis at a height h far from the plates, $h \gg L$. The electric field is weak, and the particle moves approximately in a straight line with almost constant velocity v_0 in the x direction. The Lorentz frame with this velocity (relative to the lab) is $\mathcal{F}'$ and has coordinates $(t', \boldsymbol{r}')$.

(c) (6 points) For $z' \gg L$, determine the electric the electric and magnetic fields as a function of space and time in $\mathcal{F}'$, i.e. $\boldsymbol{E}'(t', x', z')$ and $\boldsymbol{B}'(t', x', z')$. Explicitly show the dependence on the primed coordinates.

(d) (5 points) Determine the particle's acceleration in $\mathcal{F}'$ when the particle is ultra-relativistic (with v_0/c nearly one). What is the time-averaged power radiated by the charged particle in $\mathcal{F}'$?

Solution:

(a) We use separation of variables looking for trial solution

$$\phi(x, z) = X(x)Z(z)\,, \tag{2}$$

Separating variables we have from $(-\nabla^2\phi)/\phi = 0$

$$\frac{1}{X}\frac{-d^2X}{dx^2} + \frac{1}{Z}\frac{-d^2Z}{dz^2} = 0\,. \tag{3}$$

or

$$\frac{d^2X}{dx^2} = -k^2X\,, \qquad \frac{d^2Z}{dz^2} = k^2Z\,, \tag{4}$$

with k constant. Given the periodicity of in x we must have

$$X = A_n\cos(k_nx) + B_n\sin(k_nx) \qquad k_n = \frac{2\pi n}{2L}\,. \tag{5}$$

Given that the boundary conditions at $z = 0$ are odd we can limit ourselves to the sin terms with n odd. The solutions in z are $e^{\pm k_nz}$ and we limit ourselves to the decreasing solution. Putting together the ingredients the solution is

$$\phi = \sum_{n\in\text{odd}} C_n\sin(k_nx)e^{-k_nz}\,. \tag{6}$$

To determine the coefficients we match the boundary conditions at $z = 0$. Using

$$\int_0^{2L}\sin(k_nz)\sin(k_mz) = \frac{2L}{2}\delta_{nm}\,, \tag{7}$$

and the boundary condition at $z = 0$

$$\phi = V_0(1 - 2\theta(L))\,. \tag{8}$$

We have for n an odd integer

$$C_n = \frac{2}{L}\int_0^L V_0\sin(n\pi x/L)dx\,, \tag{9}$$

$$= \frac{4V_0}{\pi n}\,. \tag{10}$$

The final resul is then

$$\phi(x, z) = \sum_{n\in\text{odd}}\frac{4V_0}{\pi n}\sin(n\pi x/L)e^{-n\pi z/L}\,. \tag{11}$$

(b) At large z only the first term in the sum is dominant leading to

$$\phi = \frac{4V_0}{\pi}\sin(\pi x/L)e^{-\pi z/L} \tag{12}$$

(c) We first find the electric field in the lab frame

$$E_x = -\,\partial_x\phi = -\frac{4V_0}{L}\cos(\pi x/L)e^{-\pi z/L}\,, \tag{13}$$

$$E_z = -\,\partial_z\phi = \frac{4V_0}{L}\sin(\pi x/L)e^{-\pi z/L}\,. \tag{14}$$

Then we compute fields in the new frame using a Lorentz transformation. The Lorentz transformation rules are

$$\underline{F}^{\mu\nu}(\underline{X}) = \Lambda^\mu_\rho\Lambda^\nu_\sigma F^{\mu\nu}(X)\,, \tag{15}$$

Or in non-covariant form

$$\underline{\boldsymbol{E}}_\perp = \gamma\boldsymbol{E}_\perp - \gamma\boldsymbol{\beta}\times\boldsymbol{B}_\perp\,, \tag{16}$$

$$\underline{\boldsymbol{B}}_\perp = -\,\gamma\boldsymbol{\beta}\times\boldsymbol{E}_\perp + \gamma\boldsymbol{B}_\perp\,, \tag{17}$$

$$\underline{E}_\parallel = E_\parallel\,, \tag{18}$$

$$\underline{B}_\parallel = B_\parallel\,. \tag{19}$$

The coordinates $X = (ct, x)$ and $\underline{X} = (c\underline{t}, \underline{x})$ are related as follows

$$\underline{X}^\mu = \Lambda^\mu_\rho X^\rho\,, \tag{20}$$

or

$$X^\mu = (\Lambda^{-1})^\mu_\rho \underline{X}^\rho\,. \tag{21}$$

So if the lab frame coordinates are (t, x), the particle frame coordinates are

$$t = \gamma_0 c\underline{t} + \gamma_0 v_0\underline{x}\,, \tag{22}$$

$$x = \gamma_0 v_0\underline{t} + \gamma_0\underline{x}\,, \tag{23}$$

$$z = \underline{z}\,. \tag{24}$$

Putting together the ingredients we have

$$E^x = -\,\frac{4V_0}{L}\cos(\pi\gamma\,(v_0\underline{t} + \underline{x})/L)\,e^{-\pi\underline{z}/L}\,, \tag{25}$$

$$E^z = \gamma\frac{4V_0}{L}\cos(\pi\gamma(v_0\underline{t} + \underline{x})/L)\,e^{-\pi\underline{z}/L}\,, \tag{26}$$

$$B^y = \gamma\frac{4V_0}{L}\cos(\pi\gamma(v_0\underline{t} + \underline{x})/L)\,e^{-\pi\underline{z}/L}\,. \tag{27}$$

Discussion: For large γ, the x components can be neglected and the electric and magnetic fields are transverse. The field configuration is that of a left moving plane wave of light. The *time averaged* Poynting flux is

$$S \equiv \frac{c}{2}\left(\frac{\gamma 4V_0 e^{-\pi h/L}}{L}\right)^2 \tag{28}$$

The direction of the Poynting flux is in the negative x direction.

(d) The particle is at $x = 0$ and is moving slowly in this frame, so the $\boldsymbol{v} \times \boldsymbol{B}$ part of the Lorentz force can be neglected. The acceleration is simply $\boldsymbol{a} = q\boldsymbol{E}/m$. So

$$\boldsymbol{a} = \gamma q \frac{4V_0}{Lm} e^{-\pi h/L} \cos(\pi \gamma v_0 \underline{t}/L)\, \hat{\boldsymbol{z}} \tag{29}$$

The radiated power is simply given by the Larmour formula

$$P = \frac{2}{3}\left(\frac{q^2}{4\pi}\right)\frac{\overline{a^2}}{c^3} \tag{30}$$

Putting together the ingredients we have finally

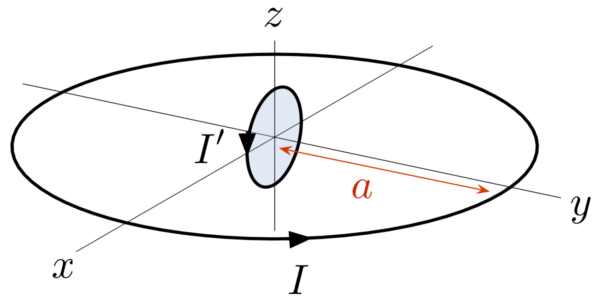
$$P = \left[\frac{8\pi}{3}\left(\frac{q^2}{4\pi mc^2}\right)^2\right] S \tag{31}$$

The term in square brackets is easily identified as the Thomson cross section of the charged particle. Thus the total power radiated is simply the cross section times the total energy flux.

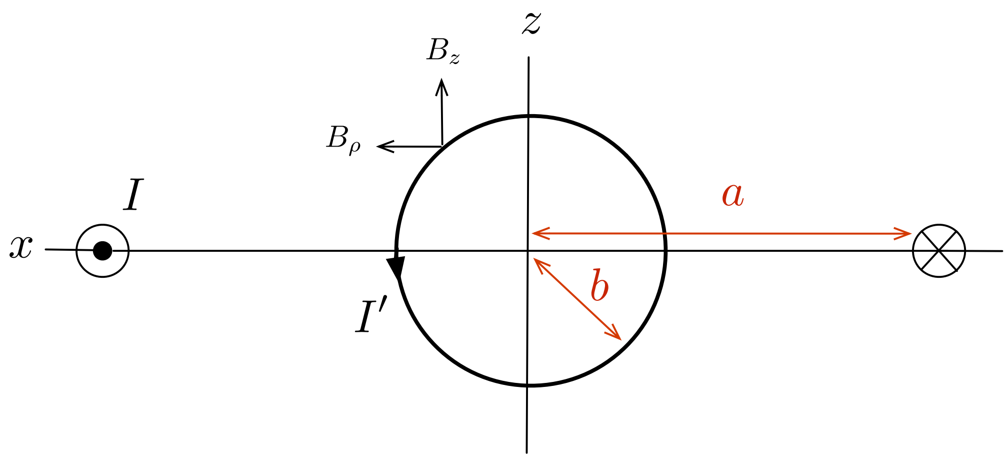
Electromagnetism 3

Two loops

Consider a pair of conducting loops centered at the origin as shown below. The large loop (loop A) lies in the xy-plane, has radius a, and carries current I. The small loop (loop B) lies in the xz-plane, has radius b, and carries current I'. Loop B is free to rotate around the x axis.



A schematic of the xz plane and the cylindrical components of $\vec{B}$ is shown below[1].



(a) (2 points) What is the magnetic field at the center of loop A?

(b) (4 points) What torque is needed to keep loop B at rest? Assume that $b \ll a$.

(c) (5 points) To refine the estimate of (b), find the z-component of the magnetic field due to loop A close to the origin, $B_z(\rho, z)$. (You may want to use $\vec{\nabla} \cdot \vec{B} = 0$, and a Taylor expansion.)

(d) (5 points) To refine the estimate of (b), find the ρ-component of the magnetic field due to loop A close to the origin, $B_\rho(\rho, z)$. (You may want to use $\vec{\nabla} \times \vec{B} = 0$, and a Taylor expansion)

(e) (4 points) At next to leading order in b/a, what torque is needed to keep loop B at rest?

[1] Here $\rho = \sqrt{x^2 + y^2}$ denotes the cylindrical radius.

Solution

(a) The magnetic field at the center of the loop is given by the Biot-Savart law

$$\vec{B}(0) = \frac{1}{c}\int \frac{\vec{I}\times d\vec{l}}{a^2} = \frac{2\pi a I}{ca^2}\hat{z} = \frac{2\pi I}{ca}\hat{z}\,. \tag{1}$$

(b) To keep the loop b at rest we need to apply a torque

$$\vec{\tau} = \vec{\mu}\times\vec{B}(0) = \left(\frac{1}{c}\pi b^2 I'\,\hat{y}\right)\times\vec{B}(0) = \frac{2\pi I I' b^2}{c^2 a}\hat{x}\,, \tag{2}$$

where $\vec{\mu}$ is the magnetic moment of a small loop of current.

(c) To refine the estimate in (2), we need to evaluate more accurately the $\vec{B}$ field across the small loop of radius b induced by the large loop of radius a. For that we need the mechanical torque relation

$$\vec{\tau} = \int \vec{r}\times\frac{1}{c}(I' d\vec{l'}\times\vec{B})\,, \tag{3}$$

To evaluate the integral we parametrize a point in loop b in cylindrical coordinates $(\hat{r},\hat{\phi})$ with ϕ the angle with the z-axis,

$$\vec{r} = b\hat{r} \qquad d\vec{l'} = bd\phi\hat{\phi} \qquad \vec{B} = B_z\hat{z} + B_\rho\hat{\rho} \tag{4}$$

with $\hat{\rho}$ the radial unit vector for cylindrical coordinates for the lop a in the xy-plane, so that the net torque along the x-direction τ_x is

$$\tau_x = \frac{I' b^2}{c}\int_0^{2\pi} d\phi\,\cos\phi(\cos\phi B_z + \sin\phi B_\rho)\,. \tag{5}$$

We need to evaluate B_z and B_ρ due to loop a. For that, we first note that since $\nabla\cdot B = 0$, Gauss law applied with a small cylindrical box of height dz and radius ρ in the xy-plane around the origin gives

$$\begin{aligned} 0 = \int_{\partial box} d\vec{S}\cdot B \approx &\quad \pi\rho^2(B_z(0,z+dz) - B_z(0,z) + 2\pi\rho dz B_\rho(\rho,z) \\ \approx &\quad \pi\rho^2 dz\frac{\partial B_z(0,z)}{\partial z} + 2\pi\rho dz B_\rho(\rho,z) \end{aligned} \tag{6}$$

which ties B_z to B_ρ,

$$B_\rho(\rho,z) \approx -\frac{\rho}{2}\frac{\partial B_z(0,z)}{\partial z}\,. \tag{7}$$

Now, we second note that near loop b we have $\vec{\nabla}\times\vec{B} = 0$, which in cylindrical coordinates means

$$\frac{\partial B_z(\rho,z)}{\partial\rho} = \frac{\partial B_r(\rho,z)}{\partial z} \approx -\frac{\rho}{2}\frac{\partial^2 B_z(0,z)}{\partial z^2} \tag{8}$$

after using (7). Along the z-axis, $B_z(0,z)$ due to loop a is readily obtained using the Biot-Savart law

$$B_z(0,z) = \frac{1}{c}\int \left(\frac{\vec{I}\times d\vec{l}}{r^2}\right)_z = \frac{2\pi a^2 I}{c(a^2+z^2)^{\frac{3}{2}}} \approx \frac{2\pi I}{ca}\left(1-\frac{3z^2}{2a^2}\right). \tag{9}$$

(d) Inserting (9) into (7-8) gives the magnetic fields near the b-loop

$$\begin{aligned} B_\rho(\rho,z) &\approx \frac{3\pi I\rho z}{ca^3} = \frac{3\pi b^2 I \sin\phi\cos\phi}{ca^3}, \\ B_z(\rho,z) &\approx \frac{2\pi I}{ca}\left(1-\frac{3z^2}{2a^2}\right) + \frac{3\pi I\rho^2}{2ca^3} = \frac{2\pi I}{ca}\left(1-\frac{3b^2\cos^2\phi}{2a^2}\right) + \frac{3\pi b^2 I\sin^2\phi}{2ca^3}. \end{aligned} \tag{10}$$

where we expressed the coordination (ρ, z) on the b loop using $\rho = b\sin\phi$ and $z = b\cos\phi$.

(e) The refined torque at next to leading order is finally

$$\begin{aligned} \tau_x \approx & \quad \frac{\pi b^2 I I'}{ac^2}\int_0^{2\pi} d\phi\left(2\cos^2\phi - \frac{3b^2\cos^4\phi}{a^2} + 3\left(\frac{1}{2}+1\right)\frac{b^2\cos^2\phi\sin^2\phi}{2a^2}\right), \\ = & \quad \frac{2\pi^2 b^2 I I'}{ac^2}\left(1-\frac{9b^2}{16a^2}\right). \end{aligned} \tag{11}$$

Quantum Mechanics 1

Quantum Mechanics (Moving wall)

A one-dimensional quantum particle of mass m is initially in the bound state of an attractive δ-functional potential $-(\hbar^2\kappa/m)\delta(x)$, where κ is a positive constant. A hard (impenetrable) wall is being moved very slowly from $x = -\infty$ towards the well, i.e. the total potential that acts on the particle is

$$U(x) = \begin{cases} -(\hbar^2\kappa/m)\delta(x)\,, & x > -R\,, \\ \infty\,, & x < -R\,. \end{cases}$$

where $-R$ is the very slowly time dependent position of the wall, $R > 0$. Everywhere in this problem, you can use the basic "zero-order" adiabatic approximation, neglecting any corrections to it.

(a) [3 points] Write down the general form of the instantaneous wavefunction $\psi(x)$ of the particle in three different regions of coordinate x: $X < -R$, $x \in [-R, 0]$, $x > 0$, and all the relevant boundary conditions $\psi(x)$ should satisfy.

(b) [3 points] From the conditions in part (a) derive the equation that determines the instantaneous energy $E(R)$ of the bound state.

(c) [4 points] Analyze the equation obtained in (b) to find the energy E_0 of the bound state of the particle without the wall, and the leading correction to E_0 when the wall is far away (R is large).

(d) [4 points] Find the distance R_e between the wall and the well at which the particle escapes from the well.

(e) [3 points] Compute the force F acting on the wall as a function of the distance R, when R is large.

(f) [3 points] Analyze as quantitatively as you can the behavior of the force for $R \to R_e$. Draw qualitatively the dependence $F(R)$ for all R.

Solution

(a) In all regions with zero potential, we have two negative-energy (unnormalized) solutions $e^{\pm qx}$ with $E = -\frac{\hbar^2 q^2}{2m}$, $q > 0$. For the bound state, $\psi(x) \to 0$ at $x \to \infty$, i.e., only e^{-qx} is admissible for $x > 0$.

For $x < -R$, where the potential is infinite, the wavefunction should vanish, $\psi(x) = 0$. Since $\psi(x)$ is continuous everywhere, $\psi(x = -R) = 0$, meaning that for $x \in [-R, 0]$, the appropriate combination of the two negative-energy solutions is $\sinh q(x + R)$. Therefore, in the two regions where it is nonvanishing, the wavefunction should have the form:

$$\psi(x) = \begin{cases} A \sinh q(x+R)\,, & x \in [-R, 0]\,, \\ Be^{-qx}\,, & x > 0\,. \end{cases}$$

Integrating Schrödinger equation over x around $x = 0$, one obtains the condition $\psi(x)$ should satisfy at the *delta*-functional potential: $\psi'(x = +0) - \psi'(x = -0) = -2\kappa\psi(x = 0)$. Together with the continuity of the wavefunction this gives the condition for the amplitudes A and B:

$$\boxed{A \sinh qR = B\,, \qquad (2\kappa - q)B = qA \cosh qR\,.}$$

(b) Dividing the first equation for the amplitudes by the second one, one obtains the equation for the wavevector q that dereminse the energy of the bound state

$$\boxed{\tanh qR = \frac{q}{2\kappa - q}\,, \qquad E(R) = -\frac{\hbar^2 q^2}{2m}\,.}$$

This equation for q can also be expressed equivalently as

$$\boxed{q = \kappa(1 - e^{-2qR})\,.}$$

(c) If R is infinitely large, from the equations in (b), we get $q = \kappa$, i.e. $E_0 = -\frac{\hbar^2\kappa^2}{2m}$. When R is large but finite, one can solve these equation by iterations. The first iteration gives

$$q = \kappa(1 - e^{-2\kappa R})\,.$$

This mean that for large R (more precisely, if $\kappa R \gg 1$) we have for the energy

$$\boxed{E(R) = -\frac{\hbar^2\kappa^2}{2m}\left(1 - 2e^{-2\kappa R}\right)\,.}$$

(d) The exact relation defining the energy of the bound state

$$q = \kappa(1 - e^{-2qR})$$

does not have a solution if R is sufficiently small. Making a sketch of both sides of this equation [or the first equation in part (b)] one sees that the plots intersect (solution exists) only if $R > R_e$, where

$$\boxed{R_e = \frac{1}{2\kappa}\,.}$$

(e) Since the energy of the particle in the bound state can change only due to the work done by the moving wall, and the force F on the wall is the opposite of the force on the particle, F is related to $E(R)$ as $-F(-dR) = dE$, i.e., we obtain it by differentiating the energy of the ground state with respect to R:

$$F(R) = \frac{dE}{dR}\,.$$

In the regime of large R,

$$F = \frac{d}{dR}\left[-\frac{\hbar^2\kappa^2}{2m}\left(1 - 2e^{-2\kappa R}\right)\right]$$

i.e.,

$$\boxed{F = -\frac{2\hbar^2\kappa^3}{m}e^{-2\kappa R}\,.}$$

(f) As we know from part (d), for $R \to R_e$, the energy of the bound state vanishes, $q \to 0$. Taking the limit $q \to 0$ in the equations of part (b), and keeping only the terms not smaller than q^2, one sees that the equations for q are reduced in this limit to

$$q = \kappa(2qR - 2(qR)^2)\,,$$

and have the solution

$$q = \frac{1}{R}(1 - \frac{1}{2\kappa R}) = \frac{1}{R^2}(R - R_e)\,.$$

This means that the energy of the bound state decreases as $(R - R_e)^2$ as $R \to R_e$, and the magnitude of the force F vanishes linearly with R: $F \propto (R - R_e)$.

Qualitative plot: the force is negative for $-\infty < R < R_e$, zero at $R > R_e$ and vanishes at $R \to -\infty$ and at $R \geq R_e$. Since $F \to 0$ as $R \to R_e$, the function $F(R)$ is continuous everywhere, and has a maximal absolute value at $R \sim \#/\kappa$.

Quantum Mechanics 2

Hard-sphere scattering

A flux J (with units $1/(\mathrm{m}^2\ \mathrm{s})$) of particles of mass m, and energy $E = \hbar^2 k^2/(2m)$ propagates in the positive z direction. The particles are scattered by the hard-wall sphere of radius a which creates a central potential $V(r)$ with the center at the origin of the coordinate system:

$$V(r) = \begin{cases} \infty\,, & r < a\,, \\ 0\,, & r > a\,. \end{cases} \tag{1}$$

This problem discusses the calculation of the s-wave total cross-section σ_0 for this scattering process.

(a) *(3 pts)* Write down the wavefunction $\psi(z)$, $z = r\cos\theta$, which describes the incident particles, and the total wavefunction $\psi(r,\theta)$ for the scattering process as a whole, where r and θ are the standard coordinates of the spherical coordinate system with the origin at the center of the potential. Normalize the incident wavefunction in a way that directly corresponds to the particle flux J.

(b) *(4 pts)* Separate out the part $\psi_0(r)$ of the incident wavefunction $\psi(z)$ which describes the particles with angular momentum $l = 0$ relative to the potential center. [*Hint:* One way of doing this is to take the average over all possible directions characterized by angle θ.]

(c) *(3 pts)* What is the general solution $\psi(r)$ of the Schrödinger equation with $l = 0$ in the region with vanishing potential (outside the sphere)?

(d) *(4 pts)* Impose the appropriate boundary condition on the total wavefunction to find the $l = 0$ scattering amplitude f.

(e) *(4 pts)* Calculate the total probability flux J_{sc} that represents the particles scattered by the scattered, that is carried by the wavefunction found above. Find then the s-wave total cross-section σ_0.

(f) *(2 pts)* Find the limit of σ_0 at low energies E of the incident particles (when the s-wave scattering is dominant) and provide a very brief (no more than 2 sentences) qualitative interpretation of σ_0 in this limit.

Solution

(a) Particles propagating with momentum $\hbar k$ in the positive z direction are described by the plane wave

$$e^{ikz}.$$

As one can check immediately, to have this wavefunction describe directly the particle flux J, the normalization factor should be $\sqrt{J/v}$, where $v = \hbar k/m$ is the particle velocity, i.e., the incident part of the wavefunction is

$$\psi(z) = \Big(\frac{mJ}{\hbar k}\Big)^{1/2} e^{ikz}.$$

Scattering by a central potential process produces the wave that is cylindrically symmetric with respect to rotation around the z-axis (i.e., independent of the polar angle ϕ) and propagating away from the center, so that the total wavefunction can be written as

$$\psi(r,\theta) = \Big(\frac{mJ}{\hbar k}\Big)^{1/2} \big(e^{ikr\cos\theta} + \frac{f(\theta)}{r} e^{ikr}\big)\,.$$

(b) To take the average over all possible directions characterized by angle θ one needs to take the following integral:

$$\langle ... \rangle = \frac{1}{2}\int_0^\pi d\theta \sin\theta ... \,.$$

Taking this integral of the incident wavefunction, one gets:

$$\psi_0(r) = \frac{1}{2}\int_0^\pi d\theta \sin\theta e^{ikr\cos\theta} = \frac{1}{2}\int_{-1}^1 dx e^{ikrx} = \frac{\sin kr}{kr}\,.$$

As should be, this expression coincides with the spherical Bessel functions $j_0(kr)$.

(c) As we know, the substitution $\psi(r) = u(r)/r$ reduces the radial part of the three-dimensional Schrödinger equation to the one-dimensional Schrödinger equation for $u(r)$. This means that for vanishing angular momentum, the general solution of the three-dimensional Schrödinger equation in the region without potential is

$$\psi(r) = \frac{1}{r}\big(A\sin(kr) + B\cos(kr)\big)\,,$$

where A and B are some constants dependent on the boundary conditions the wavefunction should satisfy.

(d) For s-wave scattering, the amplitude $f(\theta)$ in the scattered part of the wavefunction in part (a) reduces to a constant, independent of θ: $f(\theta) = f$. Comparing then the general form of the wavefunction in part (c) to the form the wavefunction should have in the s-wave scattering process as discussed in parts (a) and (b), we see that the total wavefunction should be:

$$\psi(r) = \Big(\frac{mJ}{\hbar k}\Big)^{1/2} \big(\frac{\sin kr}{kr} + \frac{f}{r} e^{ikr}\big)\,,$$

One more condition this wavefunction should satisfy for the hard-sphere scattering is that it should vanish on the surface of the sphere where the potential is infinite:

$$\psi(r = a) = 0\,.$$

This condition gives for the scattering amplitude f:

$$f = -\frac{\sin ka}{k} e^{-ika}.$$

(e) To calculate the radial part j_r of the probability flux, one needs to take the radial part d/dr of the gradient and use the standard expression for the probability flux carried by the wavefunction $\psi(r)$:

$$j_r(r) = \frac{-i\hbar}{2m}\big(\psi^* d\psi/dr - \psi d\psi^*/dr\big) = \frac{J|f|^2}{r^2}\,.$$

Since this flux is independent of the angle θ, we obtain the total scattered flux multiplying j_r with the area of a sphere $4\pi r^2$

$$I = 4\pi J|f|^2 = 4\pi J \frac{\sin^2 ka}{k^2}\,.$$

The total scattering cross-section is the ratio of the scattered flux to the incident flux:

$$\sigma_0 = J_{sc}/J = 4\pi \frac{\sin^2 ka}{k^2}\,.$$

(f) In the limit of low energies, $k \to 0$, and expression for σ_0 reduces to

$$\sigma_0 = 4\pi a^2\,.$$

We see that at low energies, solution of the Schrödinger equation is isotropic, so that the sphere scatters uniformly in all directions. This means that the scattering cross-section equals the total surface area of the sphere, not the cross-section area πa^2.

Quantum Mechanics 3

Wigner molecule

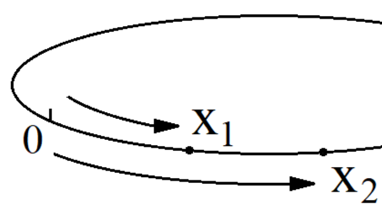
Two identical charged quantum particles with spin-1/2, mass m, and coordinates x_1 and x_2 in one dimension, interact through the Coulomb repulsion, so that the particle Hamiltonian is:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{\kappa}{|x_1 - x_2|}\,, \tag{1}$$

where $p_{1,2}$ are the particle momenta, and κ is the coefficient in the Coulomb potential.

(a) *(4 pts)* Introduce the center-of-mass coordinate R, the relative coordinate $r = x_1 - x_2$, and the corresponding momenta P and p. Make sure that the new variables satisfy canonical commutation relations. Demonstrate this explicitly. Express the Hamiltonian H in terms of the new variables. What the separation of the Hamiltonian into two independent parts means for the wavefunctions $\psi(x_1, x_2)$ of the two particles?

(b) *(5 pts)* Assume now that the particles are confined to move on a ring with a perfect circle geometry and circumference $L = 2\pi R$, i.e., the one dimensional coordinates are now measured along the circle, $x_{1,2} \in [0, L]$ as shown below.



Take the ring to be embedded in a regular three dimensional empty space, and the Coulomb interaction is now κ/r with $r = |\vec{r}_1 - \vec{r}_2|$. (i) Determine the configuration of x_1, x_2 that minimizes the interaction potential. (ii) Find the frequency ω of small oscillations around this minimum, assuming that the potential is strong, which makes it possible to adopt a quadratic approximation for the potential. (iii) What is the criterion for the validity of the harmonic approximation in the quantum mechanical problem?

(c) *(5 pts)* In this regime, strong Coulomb repulsion results also in the interaction between the spins $\vec{S}_1$ and $\vec{S}_2$ of the two electrons, producing the spin Hamiltonian

$$H_S = \lambda \vec{S}_1 \cdot \vec{S}_2\,,$$

where λ is some small energy, $\lambda \ll \hbar\omega$. What are the eigenstates and eigenenergies of this Hamiltonian?

(d) *(2 pts)* Describe the effect spin states have on the symmetry of the coordinate part $\psi(x_1, x_2)$ of the particle wavefunction.

(e) *(4 pts)* Taking into account the results of part (d), find the rotational energy spectrum of the molecule in different eigenstates of the spin Hamiltonian H_S, for the two lowest energy states of the harmonic oscillator describing the molecule bending degree of freedom r.

Solution

(a) For $R = (x_1 + x_2)/2$ and $r = x_1 - x_2$, the conjugate momenta are $P = p_1 + p_2$ and $p = (p_1 - p_2)/2$. This choice of coefficients ensures that, as should be, the coordinates and momenta satisfy the canonical commutation relations:

$$[r, p] = [R, P] = i\hbar\,, \qquad [r, P] = [R, p] = 0\,.$$

Inverting the relations between the new and the old momenta,

$$p_1 = \frac{P}{2} + p\,, \qquad p_1 = \frac{P}{2} - p\,,$$

one find the Hamiltonian in terms of the new variables:

$$H = \frac{P^2}{4m} + \frac{p^2}{m} + \frac{\kappa}{|r|}\,.$$

As usual, the effective mass for the center-of-mass motion is $2m$, while for the relative motion – $m/2$. Since the Hamiltonian

(b) To minimize the repulsion potential $V(r)$, the particles should be separated by the largest distance possible. On a circle, this means the opposite points of any diameter. In terms of the coordinate $r = x_1 - x_2$,

$$r = \frac{L}{2} \mod (L).$$

Taking into account the reduced mass $m/2$ for the relative coordinate r, one gets for the frequency of the small oscillations around this minimum:

$$\omega = \sqrt{2k/m}\,, \qquad k = V''(r)|_{r=L/2}\,.$$

If the ring is embedded in a regular empty space, the distance between the two particles on a circle can be calculated like this:

$$r = [\vec{r}_1^{\,2} + \vec{r}_2^{\,2} - 2\vec{r}_1 \cdot \vec{r}_2]^{1/2} = a[2 + 2\cos\phi]^{1/2}\,,$$

where $\vec{r}$ are the position vectors of the two particles, a is the radius of the circle ($a = L/2\pi$), and ϕ is the angle of the deviations of the relative positions of the two particles from the straight diameter, $r - L/2 = a\phi$. This gives,

$$V(r) = \frac{\kappa}{r} = \frac{\kappa}{a[2 + 2\cos\phi]^{1/2}}\,.$$

Calculating the second derivative of this expression at $\phi = 0$, we get finally

$$V''(r)|_{r=L/2} = \frac{\kappa}{8a^3}\,, \qquad \omega = \frac{1}{2a}\left(\frac{\kappa}{ma}\right)^{1/2}.$$

Quadratic approximation is legitimate, if the higher-order terms in the expansion of the potential are small. For $1/r$ potential, this condition is satisfied if the characteristics amplitude x of the oscillations is much smaller that the typical magnitude of the coordinate r, which in our case is given by a. The amplitude x of the oscillations can be estimated as the characteristic value of the oscillating coordinate in the ground state of a harmonic oscillator of frequency ω and mass $m/2$, and we get:

$$\frac{\hbar}{\omega m} \ll a^2, \quad \text{i.e.} \quad \frac{\hbar^2}{ma^2} \ll \frac{\kappa}{a}\,.$$

(c) The operator of the scalar product of the two spins, $\vec{S}_1 \cdot \vec{S}_2$, can be expressed through the operator of the magnitude of their sum $\vec{S} = \vec{S}_1 + \vec{S}_2$:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\big(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\big) = \frac{1}{2}[s(s+1) - 3/2]\,,$$

where in the second equation we took into account that $\vec{S}_1^2 = \vec{S}_2^2 = 3/4$. This equation implies that the eigenstates of the spin Hamiltonian H_S are the eigenstates $|sm\rangle$ of the total spin $\vec{S}$, and the corresponding eigenvalues depend only on its magnitude s. Addition of two spins $1/2$ produces a "triplet" of states with $s = 1$, and one "singlet" state with $s = 0$. Thus, the triplet states

$$|11\rangle = |\uparrow\uparrow\rangle\,, \quad |10\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]\,, \quad |1,-1\rangle = |\downarrow\downarrow\rangle$$

are the eigenstates of the spin Hamiltonian with the eigenenergy $\lambda/4$, while the singlet state

$$|00\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

is the eigenstate of the spin Hamiltonian with the eigenenergy $-3\lambda/4$. As usual, the arrows $|\uparrow\rangle$ and $|\downarrow\rangle$ denote here the z-components $\pm 1/2$ of the individual spins.

(d) The wavefunctions of the particles with spin $1/2$ should be antisymmetric with respect to permutation of the particles coordinates. As can be seen in part (c), the triplet eigenstates of teh sp[in Hamiltonian are symmetric while the singlet state antisymmetric with respect to the interchange of the two spins. This means that the coordinate part $\psi(x_1, x_2)$ of the wavefunction should be antisymmetric with respect to the interchange of x_1 and x_2 for the triplet spin states, i.e.,

$$\psi(x_1, x_2) = -\psi(x_2, x_1)\,,$$

and symmetric for the singlet state:

$$\psi(x_1, x_2) = \psi(x_2, x_1)\,.$$

(e) Rotational energy spectrum of the molecule is produced by its motion as a whole, described by the coordinate R. Since this motion is free, the energy of this motion is just the kinetic energy. In terms of the momentum P conjugate to R and the mass $2m$ of this motion that were obtained in part (a), this energy is given by the usual expression:

$$E = \frac{\hbar^2 P^2}{4m}\,.$$

Here the wavevector P is determined by the periodicity conditions on the molecule rotation. For the configuration with the two particles fixed at the ends of a straight diameter, one obtains the same state of the molecule, when the diameter goes through half of the full rotation, i.e., when

$$R \to R + L/2\,.$$

Note that the two identical particles of the molecule are interchanged as a result of this rotation through half of the circle. This means that the total wavefunction of the molecule should change, i.e., as we know from part (d), the coordinate part of the wavefunction should change sign in the triplet and be identical in the singlet spin states. If the bending mode of the molecule is in the ground state, the r-part of the wavefunction is preserved in this process, and only the R-part, which for free rotation, is the "plane wave" e^{iPR}, can change sign. This means that for the spin triplet, possible values of P are given by the condition

$$e^{iPL/2} = -1\,, \quad \Rightarrow \quad P = \frac{2\pi}{L}(2n+1)\,, \quad n - \text{integer}\,.$$

Therefore, for spin triplet, the rotational energy spectrum of the molecule is:

$$E_n = \frac{1}{m}\Big(\frac{\pi\hbar}{L}\Big)^2 (2n+1)^2\,.$$

For spin singlet,

$$e^{iPL/2} = 1\,, \quad \Rightarrow \quad P = \frac{2\pi}{L}(2n)\,, \quad n - \text{integer}\,,$$

and the rotational energy spectrum is:

$$E_n = \frac{1}{m}\Big(\frac{\pi\hbar}{L}\Big)^2 (2n)^2\,.$$

If the bending mode of the molecule is in the first excited state, rotation by $L/2$ inverts the direction of bending, i.e., inverts the argument of the wavefunction of this mode. Since the first excited state of the harmonic oscillator is odd, this means that the r-part of the wavefunction changes sign in the process, and the relation between the symmetry of the R-part of the wavefunction and the spin states is inverted in comparison to the situation above for the unexcited bending mode. As a result, the two rotational energy spectra found above are interchanged in this case, the odd momenta $P \propto (2n+1)$ happen in the singlet, while the even ones, $P \propto (2n)$, – in the triplet spin states.

Statistical Mechanics 1

Consider a d-dimensional electron gas, with N electrons inside a hypercubic box of length L.

(a) (2 points) Show that the number of single particle states with energy between ϵ and $\epsilon + \mathrm{d}\epsilon$ (i.e. the density of states) is

$$g(\epsilon)\,\mathrm{d}\epsilon = \alpha V \epsilon^{\frac{d}{2}-1}\mathrm{d}\epsilon \tag{1}$$

where α is a constant, and $V = L^d$ is the d-dimensional spatial volume. Determine the constant α and calculate the Fermi energy ϵ_F at zero temperature. You should leave the constant as α in what follows.

Hint: The volume of a d-dimensional sphere of radius R is $\pi^{d/2}R^d/\Gamma(d/2+1)$. Note the relation $\Gamma(x+1) = x\Gamma(x)$ and the specific values $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(b) (4 points) Calculate the zero temperature pressure and the isothermal compressibility $\kappa_T(0)$ of the d-dimensional electron gas[2]. Express your results in terms of the mean particles density $n = N/V$ and ϵ_F.

(c) (8 points) At low temperatures $T \ll \epsilon_F$ calculate the chemical potential of the d-dimensional electron gas, the mean total energy, the specific heat C_V, and the variance of energy fluctuations. You may find the Sommerfeld expansion useful:

$$\int_0^\infty \frac{H(\epsilon)}{e^{\beta(\epsilon-\mu)}+1}d\epsilon \simeq \int_0^\mu H(\epsilon)d\epsilon + \frac{\pi^2}{6\beta^2}\left.\frac{dH(\epsilon)}{d\epsilon}\right|_{\epsilon=\mu}$$

(d) (6 points) Consider the 3d case at zero temperature. A hole with area A opens on the surface of the box. (i) Calculate the electric current flowing out of the hole just after it opens. (ii) Show that the effective electrical resistance of the opening is proportional to ratio λ_F^2/A, where $\lambda_F = p_F/h$ is the Fermi wavelength.

[2] The compressibility is defined as the inverse of the bulk modulus, $\kappa_T(0)^{-1} \equiv -V\left(\partial P/\partial V\right)_{N,T=0}$

a) (2 points) We suppose our system is enclosed in a d-dimensional hypercubic box of side L and impose as a boundary condition that wavefunctions vanish at the walls. Under these conditions the allowed values of the momentum are $p = (\pi\hbar/L)n$, where n is a vector of d positive integers, in terms of which the single-particle energy is $\epsilon = |n|^2\,\pi^2\,\hbar^2/2mL^2$. The number of allowed states with energy less than or equal to ϵ is $2^{-\mathrm{d}}$ times the volume of a d-dimensional sphere of radius $\mathrm{R} = \sqrt{2mL^2\epsilon/\pi^2\hbar^2}$, which is $2\pi^{\frac{d}{2}}\,R^d/d\Gamma(\frac{d}{2})$. Thus we have

$$\mathrm{g}(\epsilon) = \frac{1}{2^{\mathrm{d}}}\frac{d}{d\epsilon}(V_d(r)) = \frac{1}{2^{\mathrm{d}}}\frac{d}{d\epsilon}\left(\frac{2\pi^{\frac{d}{2}}R^d}{d\Gamma(d/2)}\right) = cV\epsilon^{\frac{d}{2}-1}$$

Γ is the Gamma functions

$$\Gamma(x+1) = x\Gamma(x) = x\int_0^\infty \int u^{x-1}e^{-u}\,du \qquad\qquad x > 0$$

Fermi energy: $\int_0^{E_F} g(\epsilon)d\epsilon = N$

So:

$$cV\frac{2}{d}E_F^{\frac{d}{2}} = N \text{ and } E_F = \left(\frac{dN}{2cV}\right)^{\frac{2}{d}}$$

b) (4 points) An idea Fermi gas is most conveniently treated using the grand canonical ensemble. The partition function is

$$Z = \prod_{p,s}\sum_{n_{p,s}=0}^{1} e_p^{-\beta(\epsilon_{p,s}-\mu)\,n_{p,s}} = \prod_{p,s}\left(1 + e_p^{-\beta(\epsilon_{p,s}-\mu)}\right)$$

The index $\mathrm{s} = \pm\frac{1}{2}$ labels spin states. If the single-particle energy is independent of spin (and here we take $\epsilon = |p|^2/2m$ for particles of mass m), then for a gas in a container of macroscopic size the pressure and number density can be written as

$$P = \frac{kT}{V}lnZ = \frac{kT}{V}\int_0^\infty d\epsilon g(\epsilon)\ln(1 + e_p^{-\beta(\epsilon_{p,s}-\mu)})$$

$$\mathrm{n} = \frac{\langle N\rangle}{V} = \frac{1}{V}\left(\frac{\partial(\ln Z)}{\partial(\beta\mu)}\right)_{\beta,V} = \frac{1}{V}\int_0^\infty d\epsilon g(\epsilon)\frac{1}{e_p^{\beta(\epsilon-\mu)}+1}$$

At zero temperature the occupation numbers are $\mathrm{n}(\epsilon) = (e^{\beta(\epsilon-\mu)}+1)^{-1} = \theta(\epsilon_F - \epsilon)$

$$n = \frac{c}{d/2}\epsilon_F^{d/2}$$

$$P = \frac{2}{d}\frac{1}{V}\int_0^\infty d\epsilon g(\epsilon)\epsilon n(\epsilon) = \frac{c}{\left(\frac{d}{2}\right)\left(\frac{d}{2}+1\right)}\epsilon_F^{\frac{d}{2}+1} = \frac{n\epsilon_F}{\frac{d}{2}+1}$$

The isothermal compressibility can now be found as:

$$\kappa_{\mathrm{T}}(0)^{-1} = -V\left(\frac{\partial P}{\partial V}\right)_{N,T=0} = n\frac{dP}{dn} = \frac{2}{d}\left(\frac{d}{2}+1\right)P = \frac{2}{d}\epsilon_F n$$

Alternatively, the same result may be obtained by making use of the thermodynamic relation $\left(\frac{\partial\mu}{\partial N}\right)_{T,V} = \frac{V}{N^2\kappa_T}$, $\kappa_T\,(0) = \frac{V}{N^2\left(\frac{\partial N}{\partial\mu}\right)_{T=0,V}} = \frac{1}{n^2}\frac{dn}{d\epsilon_F} = \frac{d}{2n\epsilon_F}$

(c) (8 points) mean energy:

$$\langle E\rangle = -\frac{\partial\ln(Z)}{\partial\beta} \approx -\frac{\partial}{\partial\beta}\int_0^\infty d\epsilon g(\epsilon)\ln\left(1+e_p^{-\beta(\epsilon_{p,s}-\mu)}\right) = \int_0^\infty d\epsilon g(\epsilon)\frac{(\epsilon_{p,s}-\mu)}{\left(1+e_p^{\beta(\epsilon_{p,s}-\mu)}\right)}$$

$$\approx \int_0^\mu d\epsilon g(\epsilon)(\epsilon-\mu) + \frac{\pi^2}{6}\frac{1}{\beta^2}[g(\epsilon)(\epsilon-\mu)]'_\mu + \cdots =..$$

Specific heat:

$$\frac{d\langle E\rangle}{dT} \approx \frac{\pi^2}{3}k_B^2T\left[cV\epsilon^{\frac{d}{2}} - cV\epsilon^{\frac{d}{2}-1}\mu\right]'_\mu = \frac{\pi^2}{3}k_B^2T\left[c\frac{d}{2}V\mu^{\frac{d}{2}-1} - cV\left(\frac{d}{2}-1\right)\mu^{\frac{d}{2}-1}\right]$$

$$= \frac{\pi^2}{3}k_B^2TcV\mu^{\frac{d}{2}-1}$$

Energy fluctuations:

$$= -\frac{\partial^2\ln(Z)}{\partial\beta^2} \approx \frac{\pi^2}{3}\frac{1}{\beta^3}[g(\epsilon)(\epsilon-\mu)]'_\mu = \frac{\pi^2}{3}k_B^3T^3cV\mu^{\frac{d}{2}-1}$$

(d) (6 points) We can assume the opening is in the x-y plane. We need to compute the flow along the z-direction through the opening. At T=0, the electron occupation per unit volume is 2 for $p < p_F$, and zero otherwise. The number of electrons with velocity within $d^3\vec{v}$ is: $\frac{2m^3d^3\vec{v}}{h^3}$. The number of electrons with velocity $\vec{v}$ which hits the area A is then: $Av\cos(\theta)\Delta t\frac{2m^3d^3\vec{v}}{h^3}$. The induced current along the z-direction is: $dI = eAv\cos(\theta)\frac{2m^3d^3\vec{v}}{h^3}$. The total current is then:

$$I = \int eAv\cos(\theta)\frac{2d^3\vec{v}}{h^3} = \int_0^{\sqrt{\frac{2E_F}{m}}} dv\int_0^{\pi/2} d\theta\int_0^{2\pi} d\varphi 2eAv\cos(\theta)\frac{m^3v^2}{h^3}\sin(\theta)$$

$$I = \frac{2\pi eAm}{h^3}E_F^2$$

Effectively this corresponds to a resistance: $R = \frac{V}{I} = \frac{E_F}{e}\frac{h^3}{2\pi eAmE_F^2} = \frac{h^3}{2\pi e^2AmE_F}$

$$R = \frac{h^3}{\pi e^2Ap_F^2} = \frac{\lambda_F^2}{A}\frac{h}{\pi e^2}$$

Statistical Mechanics 2

Ising chain in 1d

The Hamiltonian for the Ising model in zero external magnetic field may be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1}$$

where the classical Ising spin variable $\sigma_i = \pm 1$ on each site i, and $\langle ij \rangle$ denotes nearest-neighbor pairs of sites. Consider this model in thermal equilibrium at temperature T on a one-dimensional lattice in the thermodynamic limit. Take the ferromagnetic case, $J > 0$. Derive exact expressions for the following:

(a) (10 points) the specific heat per spin C;

(b) (5 points) the spin-spin correlation function $\langle \sigma_0 \sigma_r \rangle$, where r is a position on the lattice;

(c) (5 points) and the (zero-field) magnetic susceptibility χ per spin.

Solution

The Hamiltonian for the Ising model in zero external magnetic field may be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{2}$$

where the classical Ising spin variable $\sigma_i = \pm 1$ on each site i, and $\langle ij \rangle$ denotes nearest-neighbor pairs of sites. Consider this model in thermal equilibrium at temperature T on a one-dimensional lattice in the thermodynamic limit. Take the ferromagnetic case, $J > 0$. Derive exact expressions for (i) the specific heat per spin, C; (ii) the spin-spin correlation function $\langle \sigma_0 \sigma_r \rangle$, where r denotes a position; and (iii) the (zero-field) magnetic susceptibility χ per spin.

Let $\beta = 1/(k_B T)$ and denote $\beta J \equiv K$ and the total number of sites as N. (i) The partition function for this model is

$$Z = \sum_{\sigma_n} e^{-\beta \mathcal{H}} = \sum_{\sigma_n} \prod_{ij} e^{K \sigma_i \sigma_j} \tag{3}$$

and the free energy per site is $A = -k_B T f$, where f is the dimensionless quantity

$$f = \lim_{N \to \infty} \frac{1}{N} \ln Z \tag{4}$$

Using the identity $e^{K \sigma_i \sigma_j} = \cosh K (1 + v \sigma_i \sigma_j)$, where $v \equiv \tanh K$, we can write Z as

$$Z = (\cosh K)^N \sum_{\sigma_n} \prod_{ij} (1 + v \sigma_i \sigma_j) \tag{5}$$

Quantities calculated in the thermodynamic limit do not depend on the boundary conditions, so, without loss of generality, we may take periodic boundary conditions. Then, by an explicit calculation,

$$Z = (2 \cosh K)^N (1 + v^N) = (2 \cosh K)^N + (2 \sinh K)^N \tag{6}$$

A different way to get this result is via a transfer matrix method. In a spin basis $(+, -)$ the transfer matrix $\mathcal{T}$ is

$$\mathcal{T} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \tag{7}$$

Then

$$Z = \mathrm{Tr}(\mathcal{T}^N) = \lambda_1^N + \lambda_2^N \tag{8}$$

where λ_j, $j = 1, 2$ are the eigenvalues of $\mathcal{T}$. We calculate

$$\lambda_1 = 2 \cosh K, \quad \lambda_2 = 2 \sinh K \tag{9}$$

which yields the same result as in Eq. (6). Now taking $N \to \infty$ and using the fact that $0 < v < 1$ for finite temperature, we have

$$f = \ln(\lambda_1) = \ln(2 \cosh K) \tag{10}$$

(Zero or infinite temperatures can be approached as a limit from finite temperatures.)

The internal energy per site is

$$U = -\frac{\partial f}{\partial \beta} = -J \tanh K \tag{11}$$

The specific heat per site here is

$$C = \frac{dU}{dT} = -k_B \beta^2 \frac{dU}{d\beta} = \frac{k_B K^2}{\cosh^2 K} \tag{12}$$

(ii) The spin-spin correlation function is

$$\langle \sigma_0 \sigma_r \rangle = Z^{-1} \sum_{\sigma_n} \sigma_0 \sigma_r e^{-\beta \mathcal{H}} = \frac{\sum_{\sigma_n} \sigma_0 \sigma_r \prod_{ij} (1 + v \sigma_i \sigma_j)}{\sum_{\sigma_n} \prod_{ij} (1 + v \sigma_i \sigma_j)} \tag{13}$$

An explicit evaluation yields $\langle \sigma_0 \sigma_r \rangle = v^r + v^{N-r}$. Taking $N \to \infty$ and using the fact that $0 \le v < 1$, we find

$$\langle \sigma_0 \sigma_r \rangle = v^r = (\tanh K)^r \tag{14}$$

(iii) We will present two ways to solve this problem. One way is to insert a magnetic field term in the Hamiltonian, so that $\mathcal{H} \to \mathcal{H} - H \sum_i \sigma_i$, where H denoes the external magnetic field. Denote $\beta H \equiv h$. Then calculate the transfer matrix, which is

$$\mathcal{T} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \tag{15}$$

with eigenvalues

$$\lambda_{1,2} = 2e^K \left[\cosh h \pm (\sinh^2 h + e^{-4K})^{1/2} \right] \tag{16}$$

Then we calculate $Z = \mathrm{Tr}(\mathcal{T}^N)$ and f as before, and then compute the magnetization $M(H) = \partial f / \partial h$. Finally, one calculates the zero-field susceptibility $\chi \equiv \lim_{H \to 0} \partial M / \partial H$. Following this procedure, we obtain

$$M(H) = \frac{\sinh h}{[\sinh^2 h + e^{-4K}]^{1/2}} \tag{17}$$

and

$$\chi = \beta e^{2K} \tag{18}$$

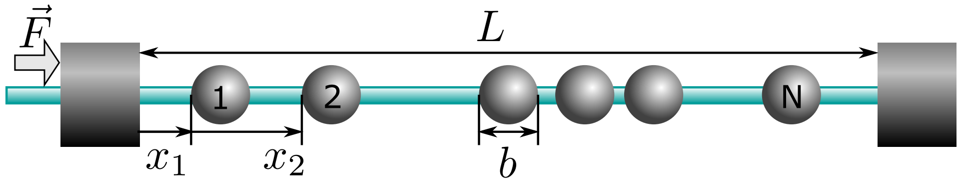
The other way to do the problem is calculate χ as the normalized sum over all spin-spin correlation functions,

$$\begin{aligned} \beta^{-1} \chi &= \sum_r \langle \sigma_0 \sigma_r \rangle \\ &= 1 + 2 \sum_{r=1}^{\infty} v^r = -1 + 2 \sum_{r=0}^{\infty} v^r \\ &= -1 + \frac{2}{1-v} = \frac{1+v}{1-v} = e^{2K} \end{aligned} \tag{19}$$

which yields the same result as in (18).

Statistical Mechanics 3

Beads on a rod



Consider $N \gg 1$ identical hard beads of diameter b and mass m moving freely on a rod of length L between two end-caps. Treat the beads as classical non-relativistic particles. All collisions between the beads are elastic and instantaneous. The internal heat capacity is c for every bead and negligible for the rod and endcaps. The beads are in thermal equilibrium at temperature T.

(a) (4pt) Write a partition function for the beads as a product of integrals, $Z = Z_p Z_x$, over their momenta p_k and coordinates x_k. *Be sure to include the limits of integration over the coordinates.*

(b) (4pt) Calculate the integral over the momenta and show that the integral over the coordinates is equal to

$$Z_x = \frac{1}{N!}[L - Nb]^N$$

Hint: it may be beneficial to change the integration variables to y_k*, where* $x_k = y_k + (k-1)b$.

(c) (4pt) Calculate the force $\vec{F}$ required to keep the endcap(s) in place if they were allowed to move as well.

(d) (4pt) Find the rate of collisions of the left bead and the endcap.

(e) (4pt) Now assume that the beads are thermally insulated, and the left endcap is slowly moved to the right, so that the rod length is decreased to $L' < L$. Find how the force F depends on L.

Solution

(a) Since each bead can move freely between collisions, their energy is

$$E = \sum_k \frac{p_k^2}{2m} \tag{1}$$

and the nearest-neighbor beads coordinates are constrained as

$$0 \le x_1 \,, \quad x_1 + b \le x_2 \,, \quad x_2 + b \le x_3 \,, \quad \ldots \,, \quad x_N \le L - b \,. \tag{2}$$

The partition function is then $Z = Z_p r Z_x$, where

$$Z_p = \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} e^{-\frac{p_k^2}{2mT}} = \left(\frac{mT}{2\pi\hbar^2}\right)^{N/2} = \lambda_T^{-N} \tag{3}$$

and

$$Z_x = \int_0^{L-Nb} dx_1 \int_{x_1+b}^{L-(N-1)b} dx_2 \int_{x_2+b}^{L-(N-2)b} dx_3 \cdots \int_{x_{N-1}+b}^{L-b} dx_N \tag{4}$$

(b) After changing the variables to $y_k = x_k - (k-1)b$ in Eq. (4), the coordinae integral becomes

$$Z_x = \int_0^{L-Nb} dy_1 \int_{y_1}^{L-Nb} dy_2 \int_{y_2}^{L-Nb} dy_3 \cdots \int_{y_{N-1}}^{L-Nb} dy_N = \int_{0\le y_1 \le \cdots \le y_N \le (L-Nb)} \prod_k dy_k \tag{5}$$

This integral is equal to $1/N!$ part of an N-dimensional hypercube with side $(L - Nb)$, and

$$Z_x = \frac{1}{N!}[L - Nb]^N \,. \tag{6}$$

This can be shown by adding all integrals with $N!$ possible permutations of y_k, which results in independent integration over each $0 \le y_k \le (L - Nb)$.

(c) The force F is the 1-dimensional equivalent of the pressure, therefore

$$F = -\left(\frac{\partial \Psi}{\partial L}\right)_T \tag{7}$$

where the free energy Ψ is (using $N \gg 1$ and Stirling's formula)

$$\begin{aligned}\Psi = -T \log Z &= -T\big[- N \log N + 1 + N \log(L - Nb) - N \log \lambda_T\big] \\ &= -NT\Big[\log\Big(\frac{L}{N} - b\Big) + 1 \log \lambda_T\Big] \,.\end{aligned} \tag{8}$$

Therefore

$$F = -\left(\frac{\partial \Psi}{\partial L}\right)_T = \frac{NT}{L - Nb} \,. \tag{9}$$

(d) The number of collisions can be calculated from the force F and the average momentum transferred to the endcap in a single collision,

$$\dot{N}_{\text{coll}} = \frac{F}{2m\overline{v}}\,, \tag{10}$$

where the average bead velocity is given by the 1-dimensional Maxwell's distribution,

$$\overline{v} = C\int_0^\infty dv\,|v|\,e^{-\frac{mv^2}{2T}} = C\frac{T}{m}\,, \tag{11}$$

and C is the normalization constant,

$$C^{-1} = \int_0^\infty dv e^{-\frac{mv^2}{2T}} = \sqrt{\frac{\pi T}{2m}}\,. \tag{12}$$

Thus, the rate of collisions is

$$\dot{N}_{\text{coll}} = \frac{N}{L - Nb}\sqrt{\frac{\pi T}{8m}}\,. \tag{13}$$

(e) The described process is completely analogous to adiabatic compression of a one-dimensional ideal gas with "volume" $L - Nb$:

$$F(L') = F(L)\left(\frac{L - Nb}{L' - Nb}\right)^\gamma\,, \tag{14}$$

where γ is the ratio of heat capacities at constant "pressure" F and "volume" L:

$$\begin{aligned}\gamma &= \frac{c_F}{c_L}\,,\\ c_L &= c + \frac{1}{2}\,,\\ c_F &= c_L + 1 = \frac{3}{2}\,.\end{aligned}$$

In the expression above, the c_L heat capacity is equal to the internal capacity c plus half-degree of freedom per equipartition theorem, and c_F is larger than c_L due to the "equation of state" (9).