STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

Classical Mechanics

August 25, 2014

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. No other materials may be used.

Classical Mechanics 1

A relativistic electron in an EM field.

a) [5 pts.] Write down the Lagrangian for a nonrelativistic point particle with charge $e = -|e|$ and mass m coupled to external, possibly time-dependent, electromagnetic potentials $A^\mu = \left(\phi, \vec{A}\right)$. Derive the Lorentz force from this Lagrangian.

b) [5 pts.] Now consider the relativistic version of a). Write down the Lagrangian. Explicitly show that the corresponding action is relativistically invariant. Check signs in your result by taking the nonrelativistic limit.

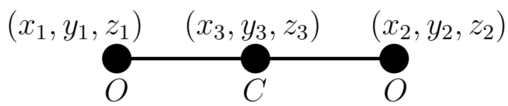
c) [5 pts.] Construct the corresponding relativistic Hamiltonian.

d) [5 pts.] Derive the equations $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ and $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$. When is the Hamiltonian for this system equal to the energy E, and when is the Hamiltonian conserved? Is the Lagrangian given by $T - V$, where T is the kinetic energy and V the potential energy? Is it conserved?

Classical Mechanics 2

Small and not so small oscillations.

Consider vibrations of the CO_2 molecule. The molecule consists of two oxygen atoms and one carbon atom which can be regarded as point particles. Electrons can be disregarded. Denote the masses of the oxygen atoms by m_O and the mass of the carbon atom by $m_C = \mu m_O$. (The value of μ is approximately $3/4$.) Denote the deviations of the oxygen atoms from their equilibrium position by (x_1, y_1, z_1) and (x_2, y_2, z_2), and those of carbon atom by (x_3, y_3, z_3). The distance between the oxygen atoms at rest is $2a$, and at rest the molecule lies along the z-axis.



a) [5 pts.] How many normal modes of oscillation with nonvanishing frequencies does this molecule have? Draw the motion of each such normal mode, but do not calculate the amplitudes of the atoms.

b) [5 pts.] Conservation of momentum and angular momentum can be used to choose a coordinate frame in which we can express all 9 coordinates in terms of a linearly independent set of coordinates. Choose such a set of coordinates, and write down the 9 relations expressing the 9 coordinates into this set. (There are many choices for this independent set, but you may choose any one.) Do not try to express the kinetic and potential energy in terms of this independent set of coordinates.

c) [5 pts.] Now consider anharmonic terms in the potential. What happens with the frequencies of the normal modes when anharmonic terms are included? As a model for a diatomic molecule we consider the following one-dimensional equation of motion (the Duffing oscillator)

$$\ddot{x} + \omega_0^2 x = \lambda x^3 \,.$$

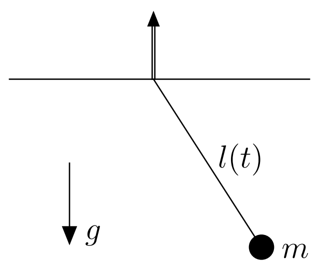
Show that naïve perturbation theory (by which we mean setting $x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \cdots$ and solving order-by-order in λ) leads to problems if we want to obtain a periodic solution.

d) [5 pts.] Indicate how these problems can be overcome.

Classical Mechanics 3

A gravitational pendulum with variable length.

The mass m of a simple gravitational pendulum is attached to a wire of length l. At time $t = 0$, the wire is pulled up slowly through a hole in the ceiling, so that the length l of the pendulum is reduced. Neglect dissipation.



a) [7 pts.] Calculate the work W that must be done to pull up the wire by a small amount dl. Show that W is independent of the (long) time interval it takes to pull up the wire.

b) [7 pts.] Calculate the change dE in the oscillation energy E of the pendulum. (By oscillation energy we mean the difference between the total energy of the pendulum and the energy of the same pendulum if it is brought to rest.) Calculate the change $d\omega$ in the oscillation frequency $\omega = 2\pi\nu$. Show that $\frac{dE}{E}$ is equal to $\frac{d\omega}{\omega}$.

c) [6 pts.] Determine the l dependence of the amplitudes for the angular and linear deviations of the mass from equilibrium.

STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

Electromagnetism

August 26, 2014

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. No other materials may be used.

Electromagnetism 1

A time dependent dipole

Consider an electric dipole at the spatial origin ($\boldsymbol{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\boldsymbol{p}(t) = p_o \cos(\omega t)\hat{\boldsymbol{z}}\,, \tag{1}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction.

a) [4 pts.] Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

b) [1 pt.] In the far field, how do the magnitude of the field strengths decrease with radius?

c) [2 pts.] Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), determine the ratio of E/B at a distance r in the far field[*]

d) [3 pts.] Estimate the total power radiated in a dipole approximation. How does this power depend on the dipole amplitude p_o, the oscillation frequency ω, and the speed of light[†]

e) [3 pts.] In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius r. (r is the distance from the origin to the observation point.)

f) [3 pts.] Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), *estimate* the ratio E/B at a distance r in the near field[‡]. Is this ratio large or small?

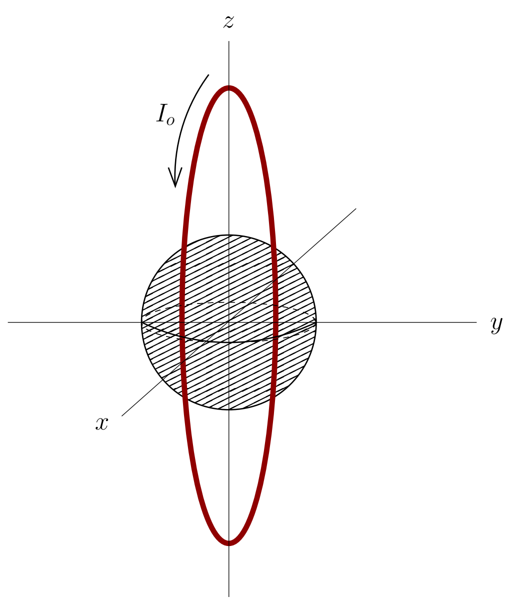
g) [4 pts.] Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

[*] In SI units this question reads, "Estimate the ratio E/cB at a distance r in the far field."
[†] In SI units this question reads, "How does the power depend on p_o, ω, c and ϵ_o?"
[‡] In SI units this question reads, "Estimate the ratio E/cB at a distance r in the near field."

Electromagnetism 2

A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



a) [5 pts.] Determine the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere.

b) [5 pts.] Determine the bound surface current on the surface of the sphere.

c) [5 pts.] What is the direction of the net-torque on the circular hoop? Indicate on the figure how the circular hoop will tend to rotate and explain your result.

d) [5 pts.] Compute the net-torque on the circular hoop.

Electromagnetism 3

EM fields of a moving charged particle

Consider a particle of charge q moving along the x-axis with a constant velocity v in such a way that time $t = 0$ when the particle is at the point $(0, 0, 0)$.

A. [6 pts.] Determine all components of the electric and magnetic fields at the point $(0, b, 0)$ in terms of q, v, t, b, the velocity of the particle $\beta = v/c$ relative to the speed of light c, and the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$.

B. [6 pts.] Show that in the highly-relativistic limit $\beta \approx 1$ and $\gamma \gg 1$, the peak transverse electric field $E_{y_{\max}}$ is

$$E_{y_{\max}} = \frac{\gamma q}{b^2} \tag{1}$$

and the peak longitudinal electric field $E_{x_{\max}}$ is

$$E_{x_{\max}} = \sqrt{\frac{4}{27}}\frac{q}{b^2} \tag{2}$$

and thus that

$$E_{y_{\max}} \gg E_{x_{\max}}. \tag{3}$$

C. [4 pts.] Show that in the highly-relativistic limit, the transverse electric field E_y is appreciable only over a time interval Δt centered on $t = 0$ given by

$$\Delta t \approx \frac{b}{\gamma v}. \tag{4}$$

D. [2 pts.] Now consider a second particle also of charge q initially at rest at the point $(0, b, 0)$. Under the "impulse approximation," the second particle is affected by the impulse produced by fields associated with the first, moving particle. Write down a condition on the mass m of the second particle in terms of the other parameters of the problem in order for the impulse approximation to be valid in the highly-relativistic limit.

E. [2 pts.] Determine the velocity of the second particle after passage of the first particle under the impulse approximation in the highly-relativistic limit.

Grad, Div, Curl, and Laplacian

CARTESIAN $\quad d\vec{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad d^3r = dxdydz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $\quad d\boldsymbol{\ell} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}} \qquad d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $\quad d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \qquad d^3r = r^2\sin\theta drd\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

Quantum Mechanics

August 27, 2014

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. No other materials may be used.

Quantum Mechanics 1

Scattering of a particle from a 3D radial potential

A particle of mass m and energy $E = \hbar^2 k^2/2m$ scatters off a 3-dimensional radial potential:

$$\begin{aligned} V(r) = -V_0 & \qquad a < r \\ 0 & \qquad r \geq a \end{aligned} \tag{1}$$

a) [4 pts.] Why does the $l = 0$ partial wave dominate the scattering near threshold (zero energy)?

b) [8 pts.] Derive an expression for the S-wave phase shift $\delta_{l=0}$ by matching at $r = a$ the $l = 0$ radial waves.

c) [8 pts.] What is the threshold cross section?

Note: It is useful to define $\hbar^2/2m(k_1^2 = k^2 + k_0^2)$ with $\hbar^2 k_0^2/2m = V_0$.

Quantum Mechanics 2

Particle with EDM moving in an electrostatic potential

Consider a particle of mass m and zero charge but an **electric dipole moment** $\vec{d} = d\vec{s}$, with $\vec{s}$ the spin of the particle. Assume that the particle moves in a spherically symmetric electro-static potential $\varphi(r)$ with $\vec{r} = (x, y, z)$

a) [4 pts.] Write down the corresponding Hamiltonian for this particle.

b) [3 pts.] Is this Hamiltonian invariant under: a) Space rotations; b) Parity; c) Time-reversal. Justify your answers.

Now assume that the particle has spin 1/2 and is confined to move between two parallel planes at $x = \pm L/2$ of a capacitor with an electric potental $\varphi(\vec{r}) = Ez$.

c) [6 pts.] Find the energies and wave functions of this particle.

d) [4 pts.] Consider the lowest energy state with momentum $p_y = 0$ and $p_z = p$. Write the corresponding wave function and the wave function you get by rotating the state an angle $\vec{\theta} = \frac{\pi}{4}\hat{x}$.

e) [3 pts.] Let $E \to E(x)$ which is slowly varying over the size of the box, i.e. $\varphi(\vec{r}) \approx (E(0) + x\partial E/\partial x + ...)z$. Calculate the change in the energy levels to first order in the small parameter $\partial E/\partial x$.

Quantum Mechanics 3

Two indistinguishable particles in a square potential well.

Consider a 1D system of two indistinguishable particles of mass m confined to an infinitely deep square potential well, $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise.

a) [4 pts.] Write down the general structure of the two-particle spatial wave function $\psi(x_1, x_2)$ and find the energy spectrum, assuming that the particles do not interact.

b) [6 pts.] Find the spatial wave function $\psi(x_1, x_2)$ for the ground state of the system. Do this for the case that the two particles each are (a) bosons with spin 0, and (b) fermions with spin-1/2. Where in the (x_1, x_2) plane are the nodes of the wave function? Explain your answer.

c) [5 pts.] Now assume that the particles are weakly interacting through the contact interaction $H' = g\delta(x_1 - x_2)$. Calculate the change to the ground-state energy to first order, again for (a) bosons with spin 0, and (b) fermions with spin-1/2. Explain your answer.

d) [5 pts.] For a system of three non-interacting spin-1/2 particles, what are the energies of the ground state and first excited state?

Grad, Div, Curl, and Laplacian

CARTESIAN $d\boldsymbol{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ $\quad d^3r = dxdydz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $d\boldsymbol{\ell} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$ $\quad d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$ $\quad d^3r = r^2\sin\theta drd\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

Statistical Mechanics

August 28, 2014

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. No other materials may be used.

Statistical Mechanics 1

Spinless fermions in degenerate energy levels.

Consider a system with two energy levels, one with energy 0 and the other with energy $\Delta > 0$. Both levels are N-fold degenerate, and the system is in equilibrium at temperature T. There are N non-interacting and effectively spinless fermions in the system.

a) [6 pts.] Assume the grand canonical ensemble with chemical potential μ to describe the system. Write down the condition that determines μ, solve it for μ, and find the occupation probabilities f and g of the upper and lower energy levels, respectively.

b) [6 pts.] Now, describe the system using the canonical ensemble. Write down the partition function and find the occupation probabilities f and g in the thermodynamic limit $N \to \infty$ [Hint: $n! \approx (n/e)^n$].

c) [4 pts.] Also within the canonical ensemble, find f and g in the low-temperature limit $T \to 0$. Compare the results to part (a) in the same low-temperature limit.

d) [4 pts.] Use the previous results to find the condition of applicability of the grand canonical ensemble to the system with a fixed number of particles N.

Statistical Mechanics 2

Magnetic system in a fixed magnetic field

Consider an equilibrium magnetic system in fixed magnetic field $B = 0$. The free energy $G(m, T)$ of the system as a function of magnetization m can be written as:

$$G(m, T) = a + \frac{b}{2}m^2 + \frac{c}{4}m^4 + \frac{d}{6}m^6.$$

In some relevant range of temperatures T, the coefficients b and d can be taken to be positive constants, $b, d > 0$, while c goes through 0 at some temperature T^* in this range:

$$c(T) = c_0(T - T^*)\,, \quad c_0 > 0\,.$$

a) [7 pts.] The free energy G describes a phase transition, in which the system goes from the state with no magnetization, $m = 0$, to the magnetized state $m = m_0 \neq 0$ at some temperature T_0. Find T_0.

b) [5 pts.] Find the magnitude of the magnetization m_0 appearing at the transition temperature T_0. What is the type of this phase transition?

c) [8 pts.] Calculate the latent heat L of the transition. State qualitatively, for what direction of the temperature change, this heat is absorbed/released by the system.

Statistical Mechanics 3

Ising chain in zero magnetic field

Consider the Hamiltonian for the Ising model on a one-dimensional lattice without external magnetic field, which may be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1}$$

where the classical Ising spin variable $\sigma_i = \pm 1$ on each site i, and $\langle ij \rangle$ denotes nearest-neighbor pairs of sites. Consider this model in thermal equilibrium at temperature T in the thermodynamic limit. Take the ferromagnetic case, $J > 0$. Derive exact expressions for

a) [8 pts.] the specific heat per spin, C

b) [7 pts.] the spin-spin correlation function $\langle \sigma_0 \sigma_r \rangle$, where r denotes a lattice site.

c) [5 pts.] the (zero-field) magnetic susceptibility χ per spin