Particle injection and trapping

One of the biggest challenges in the field is how to put particles inside the high-gradient accelerating field. It is useful to divide this discussion to the case of a beam driver and laser driver. This is because beam drivers are highly relativistic: $E = \gamma m c^2$

for e^-, $mc^2 = 0.511$ MeV

$\Rightarrow$

E	γ
1 GeV	~ 2000
10 GeV	$\sim 20{,}000$

In contrast, a laser driver is modestly relativistic. We can get an estimate for the group velocity of a laser using our earlier study of linear plasma waves:

$$\beta_g^2 = \left(\frac{v_g}{c}\right)^2 = 1 - \frac{\omega_p^2}{\omega_0^2}$$

$$\gamma_g = \left(\frac{1}{1-\beta_g^2}\right)^{1/2} = \frac{\omega_0}{\omega_p}$$

e.g. for a $1\mu m$ laser ($\omega_0 = 1.88\times10^{15}$ Hz) & $n_0 = 1\times10^{18}\ cm^{-3}$ ($\omega_p = 5.63\times10^{13}$), $\boxed{\gamma_g \sim 33}$. The nonlinear effects of high laser amplitude tend to increase γ_g, while 3D effects reduce it. For a more thorough discussion, see Esarey, Review of modern physics, 81, 1229 (2009), but the upshot is that the plasma wakefield having an equal phase velocity to the group velocity of the laser, will have a $\gamma \sim O(10)$, which means that it can be outrun by e^- of modest energy $\sim O(100$ MeV$)$ $\gamma \sim 200$.

what about β? $\beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow \beta =$

	$\gamma = 10$	$\gamma = 30$
$\beta =$	0.995	0.9994

$\beta \approx 1$ is satisfied to a very good approximation.

In what follows we will look at motion of particles in wake assuming that wake now moves w/ $\gamma_\phi = \gamma_g$

Particle orbits

Recall the constant of motion obtained from the co-moving coordinate in regular SI units:

$$\gamma m c^2 - P_z c + q(\phi - A_z c) = \text{constant}$$

This results was obtained with the co-moving coordinate defined as

$$\xi = ct - z$$

To investigate the dynamics of particles in a wakefield that is traveling at the phase velocity v_ϕ, it is more useful to define

$$\xi = v_\phi t - z \quad \dots \quad (1)$$

This is particularly useful in the case of a laser driven wakefield, since the phase velocity of wake is usually relatively small (typically, $\gamma_\phi \sim 10$)

Since the co-moving coordinate is keeping up with the wake, the quasi-static approximation is applied once again, and we obtain a constant of motion similar to the previous case, except that $c \to v_\phi$:

$$\gamma m c^2 - P_z v_\phi + q(\phi - A_z v_\phi) = \text{constant} \quad \dots \quad (2)$$

This is the Hamiltonian for the system. This can be obtained by the canonical transformation of standard E&M Hamiltonian using a generating function

$$F_2 = P_0 x - v_p \int (P_z - eA_z)\, dt$$

For an e^- & in normalized units, Eqn 2 becomes

$$\gamma - P_z \beta_\phi - \psi = \text{const} = h_0 \quad \dots \quad (3)$$

initial Hamiltonian of the particle

Here, $\psi = \frac{e}{mc^2}(\phi - v_\phi A_z)$

Note that $\Delta\psi = \psi_f - \psi_i$ for a particle then is given by

$$\Delta\psi = \gamma_f - P_{zf}\beta_\phi - \gamma_i - P_{zi}\beta_\phi \quad \dots \quad (4)$$

one special case is a highly relativistic beam, $\beta_z \sim 1$

$$\Delta\psi = \gamma_f(1-\beta_\phi) - \gamma_i(1-\beta_\phi)$$

$$= (\gamma_f - \gamma_i)(1-\beta_\phi) \times \frac{(1+\beta_\phi)}{1+\beta_\phi}$$

$\beta_\phi + 1 \sim 2$

$$\Delta\psi = (\gamma_f - \gamma_i)\frac{1}{2\gamma_\phi^2}$$

$$\Rightarrow \boxed{2\gamma_\phi^2 \Delta\psi = \Delta\gamma} \quad \text{---} (5)$$

We can use this relationship to study the possible trajectories in the phase space for the wake (Esirkepov, PRL 2008)

Consider the case of a laser wakefield accelerator in 1D:

$\nabla_\perp \sim 0 \Rightarrow$ conservation of canonical momentum: $P_\perp \approx -a_\perp$

$$h_0 = \sqrt{1 + P_z^2 + P_\perp^2} - \psi(\xi) - \beta_\phi P_z$$

$$= \left(1 + P_z^2 + A_\perp^2(\xi)\right)^{1/2} - \psi(\xi) - \beta_\phi P_z \quad \text{---} (6)$$

Each unique (and not crossing!) particle trajectory is defined in phase space by a particular h_0

Rearranging for P_z,

$$P_z = \gamma_\phi^2 \left(h_0 + \psi(\xi)\right)\left\{1 \pm \beta_\phi\left[1 - \frac{1+A^2}{\gamma_\phi^2 (h_0+\psi)^2}\right]^{1/2}\right\} \quad \text{---} (7)$$

Physical solutions for P_z exist when

$$1 \geq \frac{1+A^2}{\gamma_\phi^2 (h_0+\psi)^2} \Rightarrow (h_0+\psi)^2 \geq \frac{1}{\gamma_\phi^2} + \frac{A^2}{\gamma_\phi^2} \quad \text{---} (8)$$

The +/- components join each other where

's' for separatrix

$$P_{z,s} = \beta_\phi \gamma_\phi (1+A^2)^{1/2} \quad \text{---} (9)$$

Let's take a look at the phase space $(P_z(\xi))$ for particles for a case where $\gamma_\phi = 10$.

First, we find the wake function using Eqn 1 from the

"Laser & beam coupling to plasma" notes :

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{1}{2}\left[1-\frac{1+A^2}{(1+\psi)^2}\right]=0 \quad \ldots (10)$$

The assumption used to derive this equation was $v_\phi \approx c$, $\gamma_\phi \gg 1$. One can derive a more exact expression for a wake with any γ_ϕ, but even for the case of $\gamma_\phi \sim 10$, Eqn 10 is accurate to $O(\gamma^{-2}) \sim 1\%$. Using a "wide" laser pulse with a Gaussian profile allows us to solve eqn 10 in 1D, i.e. $\frac{\partial^2\psi}{\partial\xi^2} \rightarrow \frac{d^2\psi}{d\xi^2}$ & $\nabla_\perp^2 \rightarrow 0$. For the laser,

$$A = a_0\cos(k_0\xi)\, e^{-(\xi-\xi_0)^2/\tau^2} \quad \ldots (11)$$

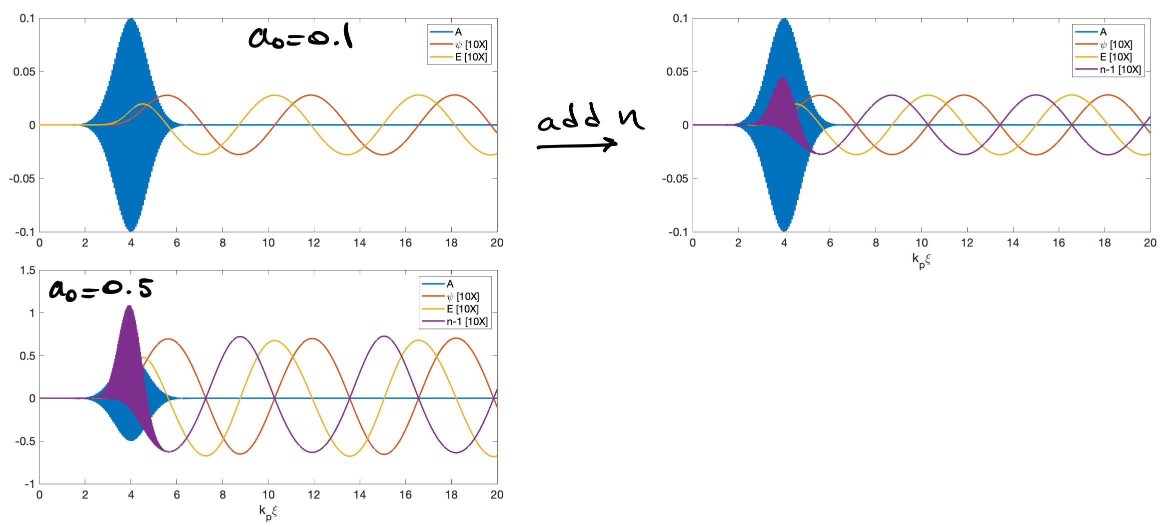
Figures below show the wake function ψ & density perturbation $(n-1)$ generated by this laser. "$n-1$" is calculated using the expressions (see "Laser & beam coupling to plasma" notes):

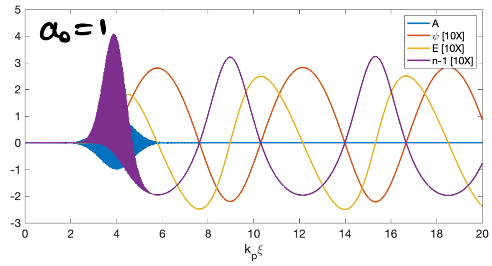
$$\frac{n}{\gamma}=\frac{1+\cancel{\nabla_\perp^2\psi}}{1+\psi} \quad \ldots (12)$$

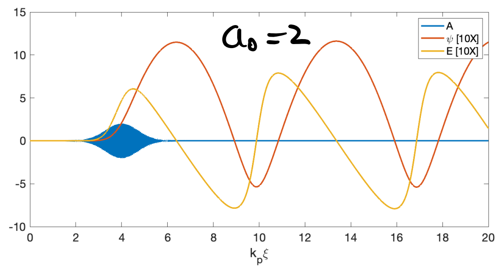
$\nabla_\perp^2\psi = 0$, 1D analysis

$P_\perp^2 = A^2$

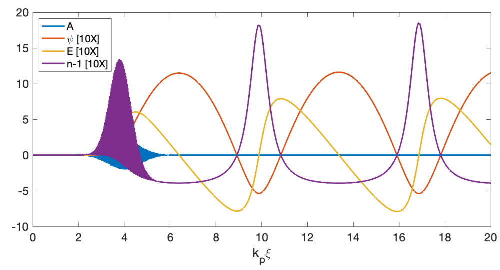
$$\gamma=\left(\frac{1+\cancel{P_\perp^2}+(1+\psi)^2}{2(1+\psi)}\right) \quad \ldots (13)$$







add n $\longrightarrow$



Note several features of these results

1. The nonlinear plasma wavelength increases with laser strength
2. In the region of the laser, density oscillates (zooming in, you can see that the density oscillations are at the second harmonic).
3. For the wake function and the electric field in the region of the laser, the oscillations are only a small perturbation on these functions.

Having obtained the wake functions, we can initialize a number of electrons with different Hamiltonians (h_0), and using Eqn 7, look at their behavior in the phase space, P_z vs. ξ.

To make the physical meaning of eqn. 8 clear, we choose a set of Hamiltonian values that result in $P_{zs} = \beta_\phi \gamma_\phi$ in the region w/o laser, i.e. $A=0$. Consider the case of

's' for separatrix

$\gamma_\phi \sim 10 \Rightarrow \beta_\phi = 0.995\,,\quad P_{zs} \sim \gamma_\phi \sim 10$

$$(6) \xrightarrow{A=0,\,\psi=0} h_0 = \sqrt{1+P_z^2} - \beta_\phi P_z \qquad \beta_\phi = \left(1-\frac{1}{\gamma_\phi^2}\right)^{1/2}$$

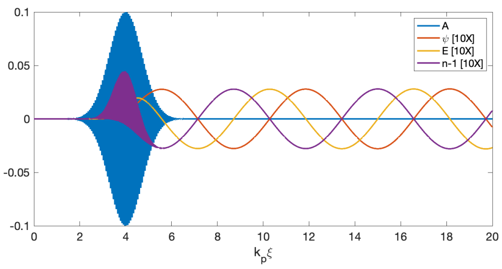
$$h_0 = P_z\left(1+\frac{1}{P_z^2}\right)^{1/2} - \beta_\phi P_z$$

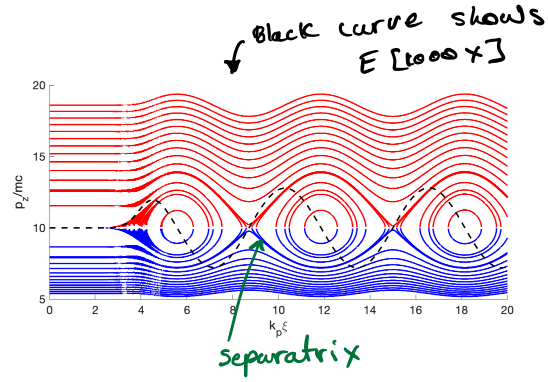
$$\approx P_z\left(1+\frac{1}{2P_z^2}\right) - P_z\left(1-\frac{1}{2\gamma_\phi^2}\right) \qquad P_z \gg 1,\ \gamma_\phi^2 \gg 1$$

$$h_0 \approx \frac{P_z}{2}\left(\frac{1}{P_z^2}+\frac{1}{\gamma_\phi^2}\right) \ldots (14) \qquad P_{z,s} = \gamma_\phi$$

$$\therefore \boxed{h_{0s} \sim 1/P_{zs}} \ldots (15)$$

e.g. for $\gamma_\phi \sim 10$, $a_0 \sim 0.1$,



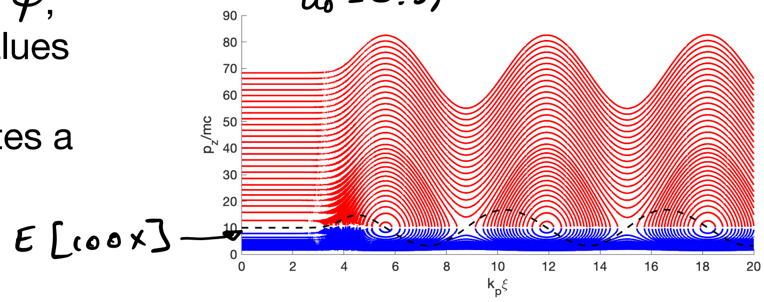


The red curve represents the positive solutions and the blue curves represent the negative solutions for Eqn 7. Note the following features:

- Trajectories that include $p_{z,s} = \gamma_\phi \beta_\phi \sim 10$, will consist of closed orbits in phase space. Note that particles with these hamiltonians do not “travel” in the wake. Mathematically, this is because the expression under the square root in equation 7 is negative for a range of ψ, and so there is no physical solution for particles in that range with a particular hamiltonian. The physical interpretation is that these particles are trapped in the wake, and they move forward along the red curve and go backwards along the blue curve in the co-moving coordinate.
- Other trajectories that not include $p_{z,s}$ simply move forward (the red curves) or backwards (blue curves). The physical interpretation for these particles is that they either have too high a momentum or too low a momentum and simply move along the wake (forwards or backwards)
- The curve separating the trapped trajectories and the traveling trajectories is called the separatrix, and represents the last “traveling particle” that isn’t trapped. This particle continually loses and gains energy but is just below the threshold of trapping.
- Note that the energy gain and loss is directly related to the electric field (the black dashed line in phase space image). Therefore, injection of prior electrons and beam loading, which modifies the electric field will distort the orbit for the electrons that haven’t been trapped yet.

What happens for a more nonlinear situation?
Now with the increased value of ψ, higher and higher momentum values can be reached for the trapped electrons as the larger field creates a larger accelerating field.

$a_0 = 0.5$,



Trapping Condition

From the phase space discussion, we know a particle is on a trapped orbit if P_z from Eqn 7 has a physical solution inside the wake for $\beta \sim \beta_\phi$. Convert $A \to P_\perp$ & rewrite eqn 8 to get

$$8 \Rightarrow (h_0 + \Psi)^2 \geq \frac{1}{\gamma_\phi^2} + \frac{A^2}{\gamma_\phi^2}$$

$$\boxed{|h_0 + \Psi| \geq \frac{\sqrt{1 + P_\perp^2}}{\gamma_\phi}} \quad \ldots (16)$$

This is the 3D trapping condition.

Note that there is a second condition. If h_0 is so large, so that the inequality in eqn 16 is satisfied regardless of Ψ, then that describes a travelling particle. Here, we require h_0 to be on the order of Ψ, so that for part of the cycle the inequality 16 is not satisfied. Only then does the Hamiltonian "h_0" describe that of a trapped particle.

Note: Trapping threshold can also be obtained by using the eqn. of constant of motion for Hamiltonian & letting $\beta_z \to \beta_\phi$

$$\gamma(1 - \beta_\phi \beta_z) = h_0 + \Psi$$

$$\text{in 1D:} \quad \gamma_\phi (1 - \beta_\phi^2) = h_0 + \Psi$$

$$\Rightarrow (h_0 + \Psi) = \frac{1}{\gamma_\phi} \quad \ldots (17)$$

Eqn 17 can be derived from 16 by setting $P_\perp \to 0$

In 3D, need to consider transverse dynamics, i.e. e^- must not transversely escape the bubble before reaching β_ϕ

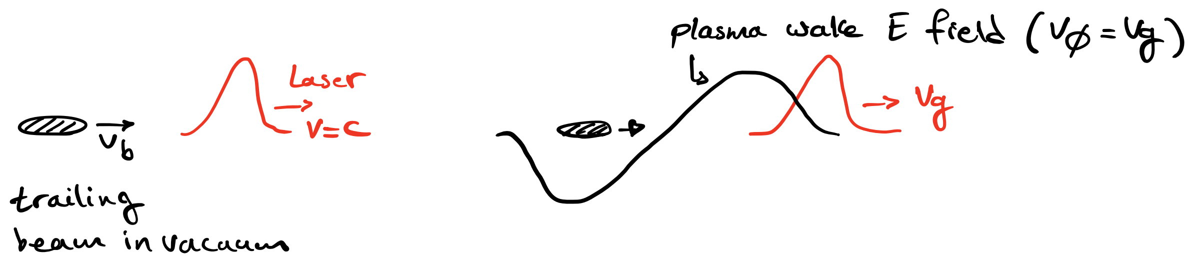
Given the equation for the Hamiltonian, we can further simplify inequality 16:

$$\left.\begin{aligned} h_0 &= \gamma_i - P_{zi} - \Psi_i \\ 16 \Rightarrow |h_0 + \Psi| &\geq \frac{\sqrt{1+P_\perp^2}}{\gamma_\phi} \end{aligned}\right\} \Rightarrow |\Delta\Psi + (\gamma_i - P_{zi})| \geq \frac{\gamma_\perp}{\gamma_\phi} \quad \ldots (18)$$

Here, $\Delta\Psi = \Psi - \Psi_i$ represents the change from the particle's initial wake function. The various trapping mechanisms then can be described in terms of inequality 18.

<u>Trapping Mechanisms</u>

There are two general classes of solutions to the problem of injecting a trailing beam on a trapped orbit. The first class is called "external injection", where a trailing beam is prepared and sent together with the driver into the plasma. Once the plasma wake forms, the trailing beam finds itself on the trapped orbit and accelerates forward:



This is a difficult task because the small size of plasma wavelength means that the trailing beam has to have a short duration $\sigma_z \lesssim \frac{1}{2} c/\omega_p$, & be focused to an equally small size. Moreover, if the driver is Laser, the jitter (meaning shot to shot variation) of laser & e-beam has to be smaller than c/ω_p.

This is a tall order, even for the most advanced e-beam sources in the world, such as LCLS, which has a jitter $\sim 20\,fs$. For an e-beam driver, this problem was circumvented by creating both driver & trailing from the same e-beam. This effectively elliminated jitter between drive & trailing beams & lead to a successful external injection experiment. See Litos, Nature, 515, 92 (2014).

The trapping condition for this mechanism is stated as follows:

$$\gamma_i \;\&\; P_{zi} \gg 1 \quad \& \quad \gamma_\perp \ll \gamma, \gamma_\phi \quad \& \quad \psi_i = 0$$

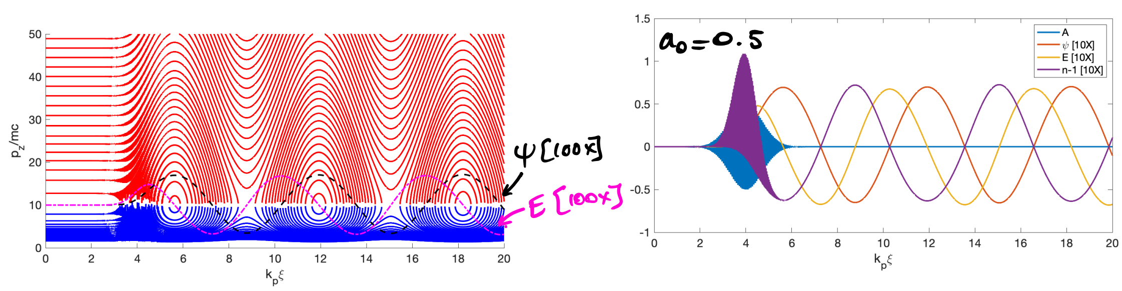
$$\boxed{\gamma^2 = 1 + P_z^2 + \cancel{P_\perp^2}}$$

$$\gamma_i - P_{zi} = \gamma_i - (\gamma_i^2 - 1)^{1/2}$$

$$= \gamma_i - \gamma_i \left(1 - \frac{1}{\gamma_i^2}\right)^{1/2}$$

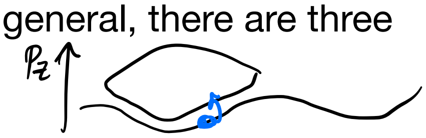
$$\gamma_i - P_{zi} = \frac{1}{2\gamma_i} \simeq \frac{1}{2P_{zi}} \quad \ldots (19)$$

$$\Rightarrow \quad 18 \Rightarrow \left|\psi + \frac{1}{2\gamma_i}\right| \geq 0 \quad \Rightarrow \quad \boxed{\psi \geq -\frac{1}{2\gamma_i}}$$



The physical interpretation is that for $\gamma_i < \gamma_\phi$, the electron may not be able to get trapped if placed at the wrong phase of the wake. In that case, it will be placed on a traveling "blue trajectory"

The second solution is to get background plasma electrons to transition from their regular passing orbits in phase space to the trapped orbits. By the way from the previous figure, you can see that if the plasma is warm enough, some electrons with $v \sim v_\phi$ will get trapped. In general, there are three strategies to facilitate this transition:

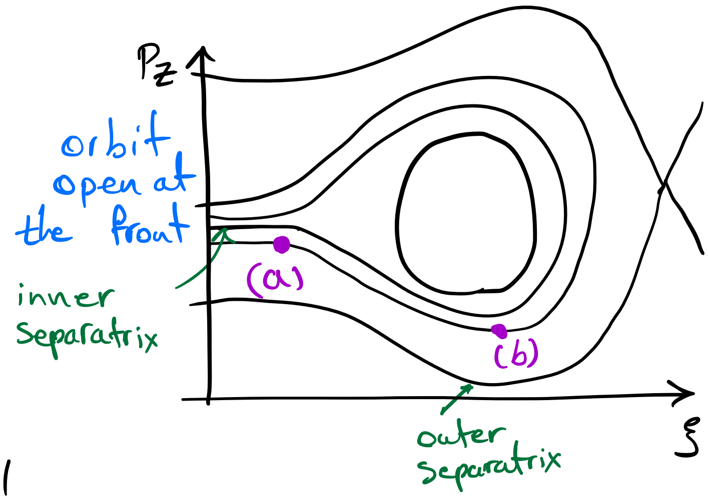


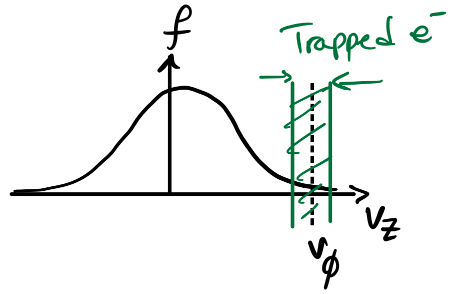
1. Initialize particles on trapped orbit

2. Sudden change in Hamiltonian
3. Drive wake to wave breaking or "self-injection" amplitude

① Initialize e^- on trapped orbit

Conceptually, this is the simplest strategy. The idea is to initialize e^- on an orbit that is already inside the seperatrix, i.e. point (a) or (b). Consider point (a). A particle in this place must have above average fluid momentum. In a thermal plasma at high temperature for example, some e^- would have $v_z \approx v_\phi$. These e^- could get trapped & accelerated provided the satisfy inequality 18





$$|\Delta\psi + (\gamma_i - P_{zi})| \geqslant \frac{\gamma_\perp}{\gamma_\phi}$$

for these e^-, $\psi_i = 0$, use inequality 19 to simplify $\gamma_i - P_{zi}$

$$\gamma_i - P_{zi} = \frac{1}{2\gamma_i}$$

$$\Rightarrow \left|\psi + \frac{1}{2\gamma_i}\right| \geqslant \frac{\gamma_\perp}{\gamma_\phi} \quad \ldots \quad (20)$$

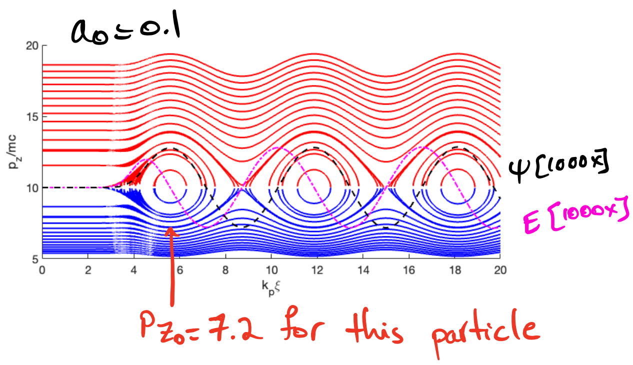
Since $\psi < 0$ at the back of the back of the wake, the minimum γ_i that will result in trapping has to satisfy

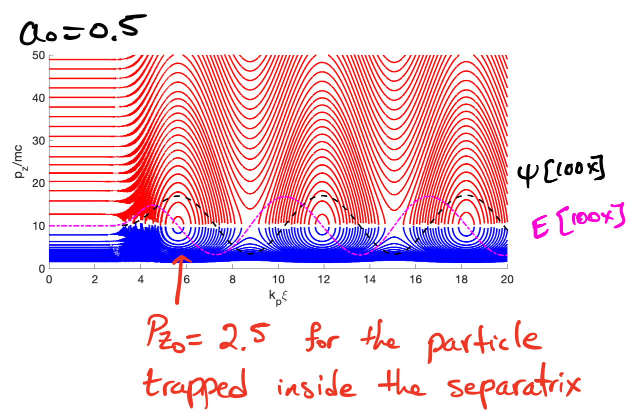
$$\psi_{min} + \frac{1}{2\gamma_i} \leqslant -\frac{\gamma_\perp}{\gamma_\phi}$$

$$\Rightarrow \frac{1}{2\gamma_i} \leqslant |\psi_{min}| - \frac{\gamma_\perp}{\gamma_\phi}$$

$$\Rightarrow \boxed{\gamma_i \geq \frac{1}{2}\left[|\varphi_{min}| - \frac{\gamma_\perp}{\gamma_\phi}\right]^{-1}} \quad \cdots (21)$$

At point (b), a particle is created within the wake (i.e. at $\varphi_i \neq 0$). From the phase space plots, it can be observed that for such a particle to be trapped in the linear regime (low a_0), it still has to be generated w/ substantial forward momentum. In the nonlinear regime, e.g. $a_0 = 2$, this gap widens to an e^- with $P_{zi} = 0$ can even get trapped.





The physical interpretation is that as the electric field amplitude increases, an electron initiated inside the wake with lowere and lower energy can gain enough energy from the wakefield to reach the phase velocity of the wake. From the trapping inequality,

$$\left|\Delta\psi + (\gamma_i - P_{zi})\right| \geq \frac{\gamma_\perp}{\gamma_\phi}$$

initiating an e^- w/ $P_{zi} \approx 0$, $\gamma_i \sim 1$ at φ_i,

$$|\Delta\psi + 1| \geq \frac{\gamma_\perp}{\gamma_\phi}$$

Since $\psi_{min} < 0$, minimum φ_i for trapping is reached for

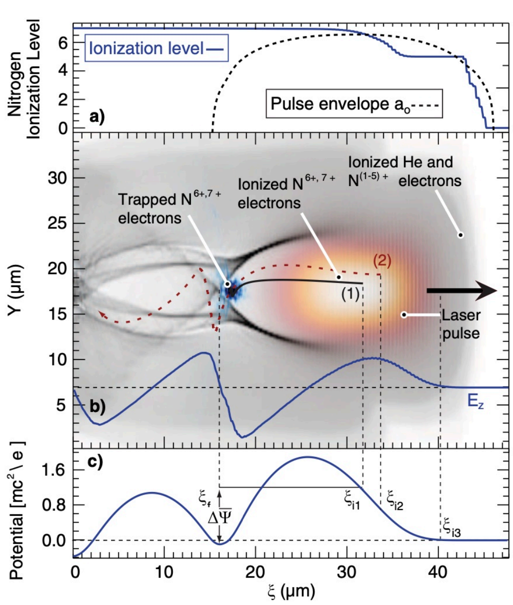
$$\Delta\psi + 1 \leq -\frac{\gamma_\perp}{\gamma_\phi}$$

$$\Delta\psi \leq -1 - \frac{\gamma_\perp}{\gamma_\phi} \quad \cdots (22)$$

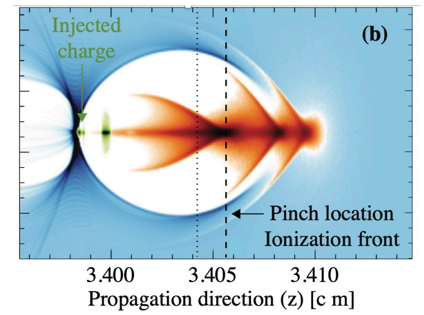
If the transverse momentum is small, $\gamma_\perp \ll \gamma_\phi$,

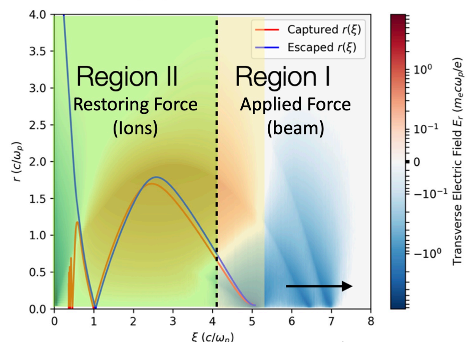
$$\boxed{\Delta\Psi \lesssim -1} \quad \cdots (23)$$

This is the condition for ionization injection, where an electron is ionized inside a wakefield at correct phase leading to its trapping. This phenomenon was first observed in electron beam experiments by Oz, et al. PRL, 2006 and by A. Pak and C. McGuffy, PRL 2010 (two back to back articles). In these experiments an inner electron shell is ionized either at the peak of the laser pulse, or at the focused point of an oscillating drive electron beam.



A. Pak, 2010





N. Vafaei-Najafabadi, 2019

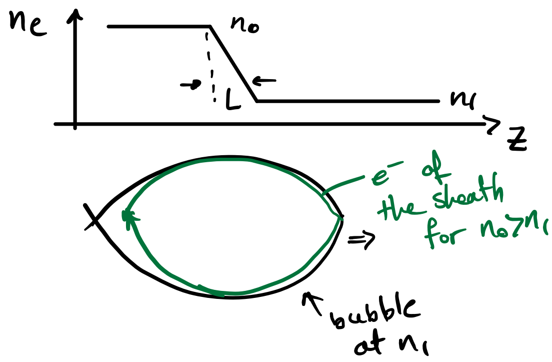
J. Yan, unpublished

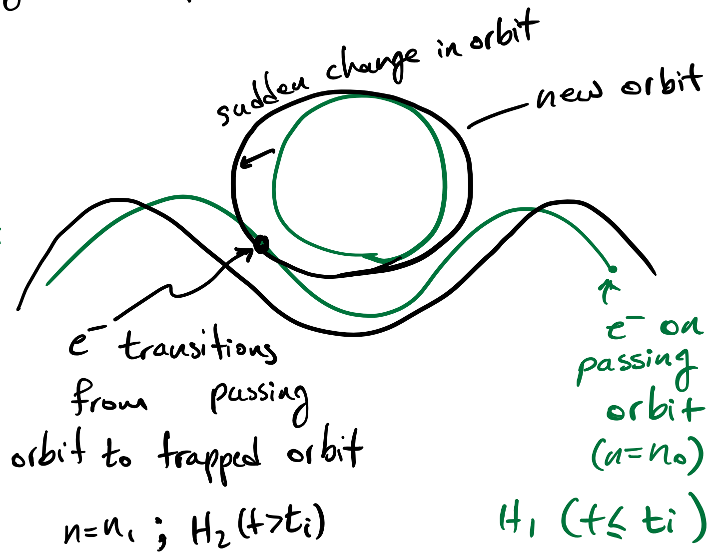
(II) Sudden Change in Hamiltonian

A sudden change in the Hamiltonian can shift the particles in phase space from a travelling orbit to a trapped orbit. These changes occur on the scale of the wake formation itself, which means that the quasi-static approximation is violated & the Hamiltonian is no longer a constant of motion.

Examples:

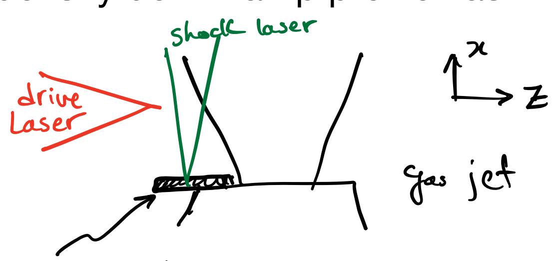
(a) sharp density downramp (also called shock injection): a change in density of plasma with a scalength comparable to c/ω_p would displace e^- from a travelling to trapped orbit:

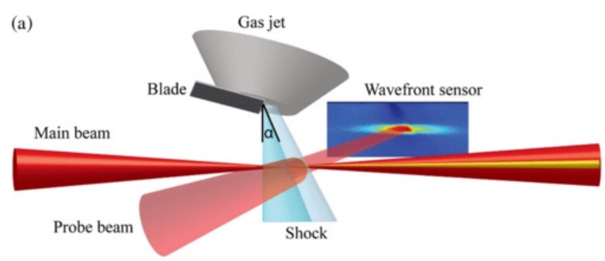




See e.g. Suk PRL 2001, Bulanov PRL 1998, Geddes PRL 2009 Scmidt PRSTAB 2012, Xu PRAB 2017

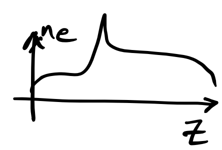
Experimentally, the most recent effort in laser wakefield to produce this density down ramp profile has involved creating a density shock



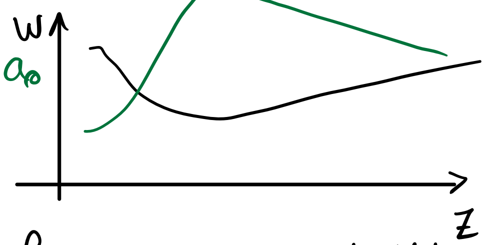


Laser-solid interaction creates a plasma plume moving in $+x$ direction. Adjusting the delay allows the driver laser to interact with shock+ gasjet (pioneered by NRL)

Another method includes partial obstruction of the gasjet, which results in a shock density profile, see e.g. K.K. Swanson, PRAB 2017



Rapid elongation of the wakefield can also naturally occur in an LWFA as a mismatched laser pulse focuses inside the plasma (see e.g. Kostyukov PRL 2009):



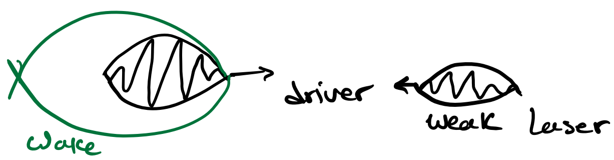
$W \downarrow$

$a_0 \uparrow$

$k_p R_b \uparrow$

Laser focusing causes bubble expansion → change in Hamiltonian.

(b) Colliding pulse injection: use a second laser to modify the wake structure at a time scale on the order of ω_p^{-1}



colliding pulses set up a standing wave & modify particle orbits. see e.g. Faure Nature 2006 & Ratchetin PRL 2009.

In general, any phenomenon that interferes with the ordinary trajectory of electrons forming the wake can lead to injection. The most commonly observed injection method in experiments is still the natural evolution of the high-amplitude plasma wake which leads to injection.

III Driving wake to wavebreaking amplitude "self injection"

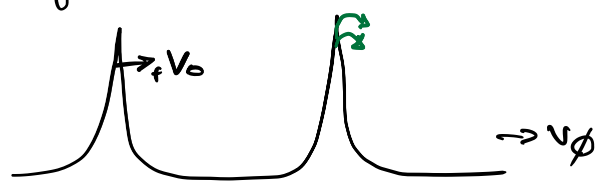
These e^- start at rest in front of the wake.

For these particles, $h_0 = \gamma - P_z - \psi = 1$ & trapping threshold is

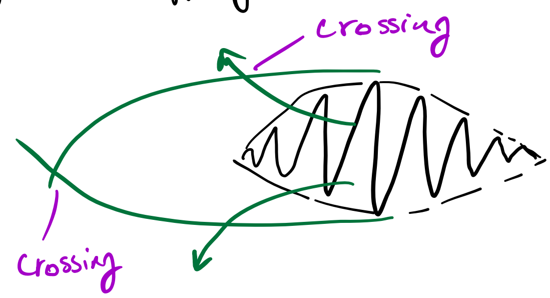
$$\Rightarrow \boxed{\psi = \frac{1}{\gamma_\phi} - 1} \quad \cdots (24)$$

The catch is that e^- have to make it to a region where $\psi \simeq -1$. This is an impossibly large value for the wake function even for highly nonlinear wakes at $a_0 \sim 4$. For these particles to get trapped then, the regular structure of the wake must "break", leading to injection of some of the background e^-

In 1D, we already saw "wave breaking" is the limit where fluid velocity v_0 approaches v_ϕ.
The wave breaks like an ocean wave & some particles roll over into the trapped orbits.



In 3D, we can have trajectory crossing w/o trapping:
But, much like the previous section, temporal variation in wake structure, can push the particles (particularly at the back of the wake structure) to become trapped.



Simulations have been used to get an empirical predictor for conditions to achieve trapping by Benedetti PoP 2013:

$$a_0(\gamma_\phi) \simeq 2.75\left(1+\left(\frac{\gamma_\phi}{22}\right)^2\right)^{1/2}$$