AMS Foundations Exam - Part A, January 2018

Name: ______________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Find the general form of the inverse of each of the following n-square matrices, or explain why this is not possible:

   (a) A lower triangular matrix with ones for all entries on and below the diagonal and zeros for all entries above the diagonal (*i.e.* $a_{ij} = 1$ if $j \leq i$, and $a_{ij} = 0$ if $j > i$).

   (b) An upper triangular matrix with ones on the diagonal and the two entries directly above the diagonal, and zeros elsewhere (*i.e.* $a_{ij} = 1$ if $i \leq j \leq i + 2$, and $a_{ij} = 0$ if $j < i$ or $j > i + 2$).

   (c) A tridiagonal matrix with ones on the superdiagonal and subdiagonal but zeros on the diagonal and everywhere else (*i.e.* $a_{ij} = 1$ if $j = i \pm 1$, and $a_{ij} = 0$ if $j \neq i \pm 1$).

2. Two matrices, $\mathbf{A}$ and $\mathbf{B}$, are said to be *simultaneously diagonalizable* if they can be diagonalized with the same set of eigenvectors (but perhaps different eigenvalues); that is, there exists some non-singular matrix $\mathbf{P}$ and diagonal matrices $\mathbf{D}_A$ and $\mathbf{D}_B$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}_A\mathbf{P}^{-1}$ and $\mathbf{B} = \mathbf{P}\mathbf{D}_B\mathbf{P}^{-1}$. Show that two diagonalizable matrices $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable if and only if they commute (*i.e.* $\mathbf{AB} = \mathbf{BA}$).

3. The Monroe-Penrose inverse (pseudoinverse), $\mathbf{A}^+ \in \mathbb{R}^{n\times m}$, of a real matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is defined as a matrix satisfying four criteria:

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \qquad \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+, \qquad \left(\mathbf{A}\mathbf{A}^+\right)^T = \mathbf{A}\mathbf{A}^+, \qquad \left(\mathbf{A}^+\mathbf{A}\right)^T = \mathbf{A}^+\mathbf{A}$$

(a) Show that, if $rank(\mathbf{A}) = m$, $\mathbf{A}^+ = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$ and $\mathbf{A}\mathbf{A}^+ = \mathbf{I}_m$.

(b) Show that, if $rank(\mathbf{A}) = n$, $\mathbf{A}^+ = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ and $\mathbf{A}^+\mathbf{A} = \mathbf{I}_n$.

4. Evaluate $f(\mathbf{A}) = e^{(-8\mathbf{A})}$, $g(\mathbf{A}) = \sin(\mathbf{A})$, and $h(\mathbf{A}) = \ln(4\mathbf{A})$, where:

$$\mathbf{A} = \frac{\pi}{8}\begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Find a general solution for the lengths of the sides of the rectangular parallelepiped with the largest volume that can be inscribed in the ellipsoid defined by $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$, where a, b, and c are positive real constants. *Definitions:* A parallelepiped is a three dimensional object with 6 sides, all of which are parallelograms; inscribed means that the boundaries touch but do not cross.

2. Consider the unnormalized three-variable Gaussian function defined by:

$$G(x,y,z) = e^{-\left(\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right)}$$

where a, b ,and c are positive real constants.

(a) Find the hypervolume lying between this surface and the x-y-z hyperplane, over the entire domain of $\mathbb{R}^3$; that is, the volume in $\mathbb{R}^4$ defined by:

$$w \in [0, G(x,y,z)], \quad x \in (-\infty,+\infty), \quad y \in (-\infty,+\infty), \quad z \in (-\infty,+\infty).$$

(b) Find a function $N(x,y,z) = kG(x,y,z)$, $k \in \mathbb{R}$, such that:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} N(x,y,z)dxdydz = 1$$

3. Find a general solution for $\int at^n e^{bt} dt$, where n is any integer, and a and b real constants.

4. Consider the set of functions $Q = \{q(x, y, z)\}$ such that:

$$q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + l$$

where a, b, c, d, e, f, g, h, k, and l are any real constants.

(a) Prove that every element of Q has at most one local maximum, minimum or saddle point.

(b) Do any elements of Q have no critical points? Explain your answer.

(c) Explain how one may characterize a given $q(x, y, z)$ as having a local maximum, minimum, saddle point, or none of these.