APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Summer 2023 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 24, 2023

Time: 9:00 AM – 1:00 PM

A1. Solve the following initial-boundary value problem using the separation of variable method

$$u_t = ku_{xx} + f(x,t), \quad x \in (0,1)$$

$$u(0,t) = \alpha(t), \quad u(1,t) = \beta(t),$$

$$u(x,0) = g(x).$$

Introduce your own notations to make the solution compact and clean.

Continue solution:

A2. Solve the normalized one dimensional wave equation

$$u_{tt} - u_{xx} = 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = \delta(x).$$

(a). For $x \in (-\infty, \infty)$.

(b). For $x \in (-2, \infty)$ and $u(-2,t) = 0$.

(c). For $x \in (-2, \infty)$ and $u_x(-2,t) = 0$.

(c). For $x \in (-2, 2)$ and $u(-2,t) = u(2,t) = 0$.

(d). For $x \in (-2, 2)$ and $u(-2,t) = u_x(2,t) = 0$.

Continue solution:

A3. Given the initial condition as

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

solve the Riemann problem for conservation law

$$u_t + f(u)_x = 0,$$

if the flux function is given as

(a). If $f(u) = u^2, \ \ u > 0.$

(b). If $f(u) = u(1-u), \ \ u > 0.$

(c). If $f(u) = u^{3/2}, \ \ u > 0.$

(d). If $f(u) = u(1-\ln u), \ \ u > 0.$

Continue solution:

A4. Consider the mapping of the complex plane given by $f(z) := \frac{z}{z+i}$. Let $\mathcal{S} := \{-1 < \mathrm{Im} < 1\}$. Describe $f(\mathcal{S})$. Please show all details of your computation.

Continue solution:

A5. Let $g(x)$ be a continuous, real-valued function that is ***zero outside*** of the interval $[-\pi, \pi]$. Define

$$G(z) = \int_{-\infty}^{\infty} f(t)\cos(zt)dt.$$

(i) Prove that $G(z)$ is an entire function of z.

(ii) Prove that g and G cannot agree on all of the real axis unless g is identically 0.

Continue solution:

B6. Polynomial interpolation is a fundamental tool in numerical analysis with broad applications.

(a) (3 points) Given $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, describe how to construct an interpolating polynomial $P(x)$ of degree at most n using Newton's divided differences.

(b) (3 points) Discuss Runge's phenomenon, where increasing the degree of the interpolating polynomial may lead to a worse approximation. Suggest a method to mitigate this issue.

(c) (4 points) Using the Lagrange form of $P(x)$, derive the weights for the quadrature rule with x_i with $0 \leq i \leq n$ as the quadrature points over an interval $[0, 1]$. What is the degree of the resulting quadrature rule if the points $\{x_i\}$ are the roots of the degree-$(n+1)$ Legendre polynomial?

Continue solution:

B7. Consider the optimization problem of finding the minimum value of a function $f : \mathbb{R}^n \to \mathbb{R}$, which is twice continuously differentiable.

(a) (3 points) Describe Newton's method for optimization, and derive the update rule in the context of minimizing f. Specifically, explain how the second-order Taylor approximation to f at a point x_k is used to derive the Newton step.

(b) (2 points) Under what conditions is Newton's method guaranteed to converge?

(c) (3 points) Consider the application of Newton's method to the Rosenbrock function:

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2.$$

Write down the formula for the first step of Newton's method starting from the initial point (0,0), and compute the resulting new point. Discuss any difficulties that might arise in this step.

(d) (2 points) Newton's method is often compared with gradient descent. Discuss one situation where Newton's method might be preferred over gradient descent, and vice versa.

Continue solution:

B8. Consider the ordinary differential equation (ODE):

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

(a) (4 points) An explicit second-order Runge-Kutta method can be expressed as:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + h, y_n + hk_1), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2). \end{aligned}$$

Perform a linear stability analysis of this method by considering the special case $y' = \lambda y$ with $\lambda \in \mathbb{C}$. It suffices to derive an inequality in terms of λh and use the inequality to explain whether the scheme is conditionally or unconditionally stable.

(b) (4 points) An implicit second-order Runge-Kutta method can be formulated as:

$$\begin{aligned} k_1 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ y_{n+1} &= y_n + hk_1. \end{aligned}$$

Again, carry out a linear stability analysis for this method as in part (a).

(c) (2 points) Discuss the differences between the two methods in terms of accuracy, stability, and computational efficiency. What types of problems are each method best suited for, and what challenges might arise when implementing these methods?

Continue solution:

B9. Consider the following IBVP:

$$
\begin{aligned}
&v_t = \nu v_{xx}, \quad x \in (0,1) \\
&v(x,0) = f(x), \quad x \in [0,1] \\
&v_x(0,t) = 0, \quad t > 0 \\
&v(1,t) = 0, \quad t > 0
\end{aligned}
$$

(a) Consider two discretizations of the Neumann boundary condition at $x = 0$: (i) one-sided, first-order finite difference using the boundary point and the 1st grid point at Δx and (ii) centered second-order finite difference using a ghost point at $-\Delta x/2$ and the 1st grid point at $\Delta x/2$. Derive the FTCS scheme and present it in a matrix form for each type of the boundary condition discretization.

(b) Analyze the norm consistency of both numerical schemes in l_∞ norm and discuss your results.

Continue solution:

B10.

(a) Describe main steps of high-resolution slope-limiter methods for hyperbolic conservation laws.

(b) Show that a slope-limiter method operating with a piecewise linear reconstruction $\tilde{u}^n(x) = u_k^n + \sigma_k(x - x_k)$ in each cell $x \in [x_{k-1/2}, x_{k+1/2}]$ and the minmod limiter

$$\sigma_k^n = minmod\left(\frac{u_k^n - u_{k-1}^n}{\Delta x}, \frac{u_{k+1}^n - u_k^n}{\Delta x}\right), \quad \text{where} \quad minmod(a,b) = \begin{cases} a, & |a| < |b|, \quad ab > 0 \\ b, & |b| < |a|, \quad ab > 0 \\ 0, & ab \leq 0, \end{cases}$$

is TVD (a graphical schematic with explanation is sufficient for the reconstruction step).

(c) Show that the method in (a) applied to the linear advection equation $v_t + av_x = 0$, $a > 0$, is equivalent to the Lax-Wendroff method if the piece-wise linear reconstruction is used and the slopes are computed as $\sigma_k^n = \frac{u_{k+1}^n - u_k^n}{\Delta x}$.

(d) Based on this selection of slopes (illustrate them graphically), explain why the Lax-Wendroff scheme is not TVD.

Continue solution: